Hwk #10:

Calculate the curvature of the helix (spiral staircase) given by $\vec{x}(t) = [r\cos t\,,\ r\sin t\,,\ ht]^T$.

Solution: There are two methods: Either way we calculate $\vec{x}\,'(t) = [-r\sin t\,,\ r\cos t\,,\ h]^T$. Therefore $\frac{ds}{dt} = \|\vec{x}\,'(t)\| = \sqrt{r^2+h^2}$.

Consequently, the unit tangent vector is $\vec{T}(t) = [-r\cos t\,,\ r\cos t\,,\ h]^T/\sqrt{r^2+h^2}$. We get $\kappa = \|d\vec{T}/ds\| = \|\vec{T}'(t)\|/\|\vec{x}\,'(t)\| = r/(r^2+h^2)$.

Alternatively, using the formula $\kappa = \|\vec{x}\,'(t)\times\vec{x}\,''(t)\|/\|\vec{x}\,'(t)\|^3$, we find

$$\vec{x}\,'(t)\times\vec{x}\,''(t) = \begin{bmatrix} -r\sin t \\ r\cos t \\ h \end{bmatrix} \times \begin{bmatrix} -r\cos t \\ -r\sin t \\ 0 \end{bmatrix} = \begin{bmatrix} rh\sin t \\ -rh\cos t \\ r^2 \end{bmatrix}$$

which has norm $r\sqrt{r^2+h^2}$.

Again we find $\kappa = r/(r^2+h^2)$.

Hwk #11:

Let $\vec{u} = [2,1,3]^T$, $\vec{v} = [-1,0,4]^T$, $\vec{w} = [2,-1,-3]^T$. Calculate $\vec{v}\times\vec{w}$, $\vec{u}\times(\vec{v}\times\vec{w})$, $\vec{u}\times\vec{v}$, $(\vec{u}\times\vec{v})\times\vec{w}$.

With these same vectors from the previous problem, calculate $\vec{u}\cdot(\vec{v}\times\vec{w})$ and $\vec{w}\cdot(\vec{u}\times\vec{v})$.

Solution:

$$\vec{v}\times\vec{w} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0(-3)-(-1)4 \\ 4\cdot 2-(-1)(-3) \\ (-1)(-1)-0\cdot 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

$$\vec{u}\times(\vec{v}\times\vec{w}) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1-3\cdot 5 \\ 3\cdot 4-2\cdot 1 \\ 2\cdot 5-1\cdot 4 \end{bmatrix} = \begin{bmatrix} -14 \\ 10 \\ 6 \end{bmatrix}$$

$$\vec{u}\times\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 4-3\cdot 0 \\ 3(-1)-2\cdot 4 \\ 2\cdot 0-1(-1) \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ 1 \end{bmatrix}$$

$$(\vec{u}\times\vec{v})\times\vec{w} = \begin{bmatrix} 4 \\ -11 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} (-11)(-3)-1(-1) \\ 1\cdot 2-4(-3) \\ 4(-1)-(-11)2 \end{bmatrix} = \begin{bmatrix} 34 \\ 14 \\ 18 \end{bmatrix}$$

(If you find it still difficult to safely remember the pattern of which to multiply with which, a quick check towards correctness of your calculation is to confirm that the dot product of the result with either factor is 0.)

Specifically our example has shown that $\vec{u}\times(\vec{v}\times\vec{w})$ and $(\vec{u}\times\vec{v})\times\vec{w}$ are different!

Now we calculate

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = 2\cdot 4+1\cdot 5+3\cdot 1 = 16$$

$$\vec{w} \cdot (\vec{u} \times \vec{v}) = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -11 \\ 1 \end{bmatrix} = 2 \cdot 4 + (-1)(-11) + (-3) \cdot 1 = 16$$

As they should, these two are equal.

Hwk #12:

Find the area of the triangle whose vertices are the points $A(1,1,3)$, $B(-2,3,0)$, $C(1,1,-2)$.

Solution: $\vec{AB} = [-3,\, 2,\, -3]^T$ and $\vec{AC} = [0,\, 0,\, -5]^T$. The area of the triangle is half the area of the parallelogram spanned by these two vectors, and this parallelogram area is the norm of the cross product of the two vectors. So the area is

$$\frac{1}{2} \left\| \begin{bmatrix} 10 \\ 15 \\ 0 \end{bmatrix} \right\| = \frac{1}{2}\sqrt{325} = \frac{5}{2}\sqrt{13}$$

Note: You could have done the same kind of calculation (eg) with $\vec{BA}$ and $\vec{BC}$. Different route, but the result would have to be the same. Try it out if you wish.

Hwk #13:

Given the vectors $\vec{u} = [u_1, u_2, u_3]^T$ and $\vec{v} = [v_1, v_2, v_3]^T$ in space, I have defined their cross product $\vec{u} \times \vec{v}$ to be the vector $\vec{w} = [u_2 v_3 - u_3 v_2\,,\; u_3 v_1 - u_1 v_3\,,\; u_1 v_2 - u_2 v_1]^T$. Show that indeed, $\|\vec{w}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2 \varphi)$, where φ is the angle between $\vec{u}$ and $\vec{v}$.

Solution: The right side is $\|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$. Now let's evaluate either side in components:

$$\begin{aligned} \|\vec{w}\|^2 &= (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2 = \\ &= u_2^2 v_3^2 + u_3^2 v_2^2 + u_3^2 v_1^2 + u_1^2 v_3^2 + u_1^2 v_2^2 + u_2^2 v_1^2 - 2u_2 u_3 v_2 v_3 - 2u_1 u_3 v_1 v_3 - 2u_1 u_2 v_1 v_2 \end{aligned}$$

$$\begin{aligned} \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\ &= u_2^2 v_3^2 + u_3^2 v_2^2 + u_3^2 v_1^2 + u_1^2 v_3^2 + u_1^2 v_2^2 + u_2^2 v_1^2 - 2u_2 u_3 v_2 v_3 - 2u_1 u_3 v_1 v_3 - 2u_1 u_2 v_1 v_2 \end{aligned}$$