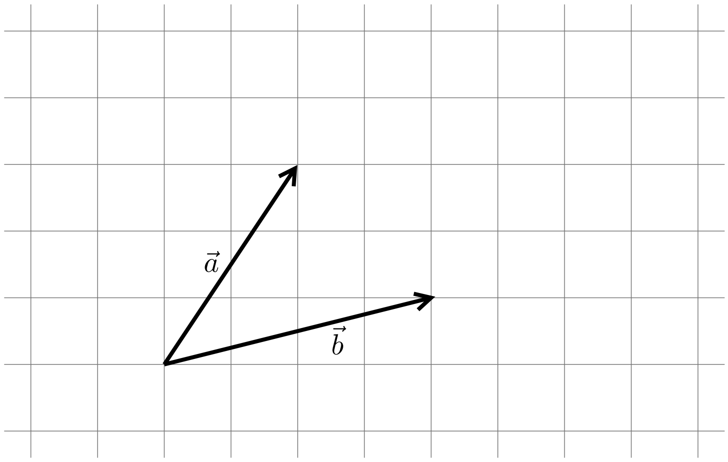
Homework

Math 247 – Honors Calculus 3

Fall 2015 – Jochen Denzler

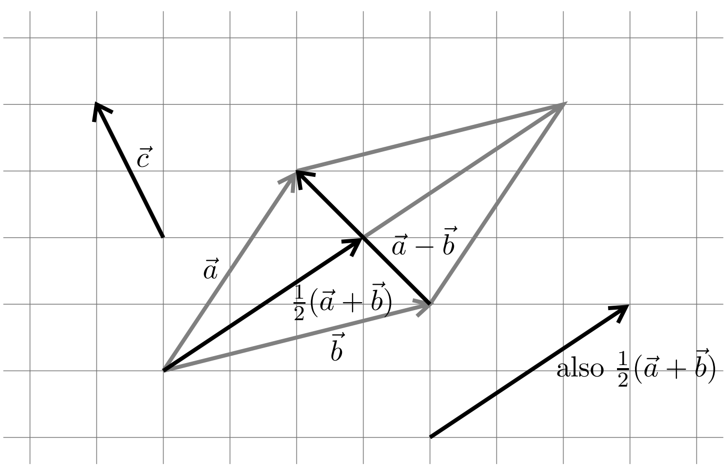
Hwk #1:

The geometric vectors $\vec{a}$ and $\vec{b}$ are as given in the figure below. The coordinate lines drawn are at unit distances apart. The vector $\vec{c}$ is given in coordinates: $\vec{c} = [-1\,,\ 2]^T$.



(a) Draw $\vec{a} - \vec{b}$ and $\frac{1}{2}(\vec{a} + \vec{b})$ into the figure.
(b) Draw $\vec{c} = [-1\,,\ 2]^T$ into the same figure.
(c) Find the coordinates of $\vec{a}$ and $\vec{b}$, and calculate their dot product.
(d) Find the angle between $\vec{a}$ and $\vec{b}$ to a numerical precision of $1/100^{\text{th}}$ of a degree. Since you cannot read off the coordinates from a picture to sufficient precision, I'll tell you that the coordinates of $\vec{a}$ and $\vec{b}$ are indeed intended to be precise integers.

Solution:



$\vec{a} = [2\,,\ 3]^T$

$\vec{b} = [4\,,\ 1]^T$

$\vec{a} \cdot \vec{b} = 2 \cdot 4 + 3 \cdot 1 = 11$

$$\cos \angle(\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \frac{11}{\sqrt{2^2 + 3^2}\,\sqrt{4^2 + 1^2}} = \frac{11}{\sqrt{13 \cdot 17}}$$

Pocket calculator: $\varphi = \arccos \frac{11}{\sqrt{221}} = 0.73782\text{rad} = 42.27°$

Note that there is no origin in this figure, and none is needed. All vectors could have been placed elsewhere, as has been done for illustration by putting a copy of $\frac{1}{2}(\vec{a} + \vec{b})$ in a second place.

Hwk #2:

(a) Show that $2\|\vec{a}\|^2 + 2\|\vec{b}\|^2 = \|\vec{a} - \vec{b}\|^2 + \|\vec{a} + \vec{b}\|^2$. Draw a figure for illustration, and see why this formula is called the parallelogram identity.

(b) The dot product can be reconstructed from the norm: Indeed, show that $\vec{x} \cdot \vec{y} = \frac{1}{4}(\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2)$.

Solution: Its easier to start evaluating these equalities from the right hand side:

$$\begin{aligned}
\|\vec{a} - \vec{b}\|^2 + \|\vec{a} + \vec{b}\|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) + (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \\
&= (\vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}) + (\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}) \\
&= 2\vec{a} \cdot \vec{a} + 2\vec{b} \cdot \vec{b} \\
&= 2(\|\vec{a}\|^2 + \|\vec{b}\|^2)
\end{aligned}$$

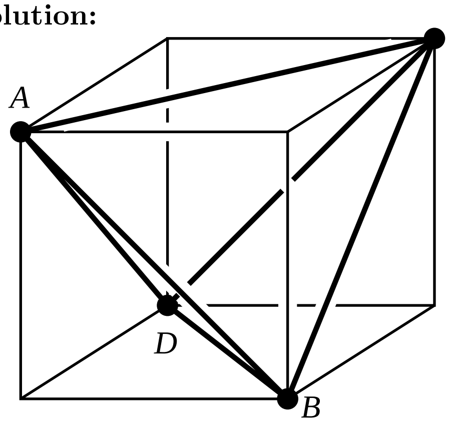
$$\begin{aligned}
\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) - (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y}) \\
&= (\vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{x} + \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}) - (\vec{x} \cdot \vec{x} - \vec{y} \cdot \vec{x} - \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}) \\
&= 2\vec{y} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} \\
&= 4\vec{x} \cdot \vec{y}
\end{aligned}$$

Hwk #3:

Let A, B, C, D be the vertices of a regular tetrahedron (a polyhedron whose four faces are congruent equilateral triangles). Let O be the center of this tetrahedron. Find the angle AOB. Interpretation: Chemists are interested in this angle b/c methane has a carbon atom in the center and a hydrogen atom at each of the vertices of the tetrahedron.

Hint: Take a cube, whose center is the origin, with the axes of a coordinate system parallel to the sides of the cube; choose four of its eight vertices in a checkerboard manner: if one vertex of the cube is chosen, then the immediately adjacent ones are not and vice versa. The chosen vertices form the corners of a tetrahedron. Draw a figure. It is easy to calculate (eg) the dot product $\vec{OA} \cdot \vec{OB}$ in terms of coordinates. The formula for this same dot product in terms of norms and angles can then be used to find the angle.

Solution:



Assume the sides to be of length 2 and the origin O in the center of the cube, the x axis horizontal, the z axis vertical in the drawing plane, and the y axis perpendicular into the drawing plane. Then the coordinates are as follows:
$A = (-1, -1, 1)$, hence $\vec{a} := \vec{OA} = [-1, -1, 1]^T$
$B = (1, -1, -1)$, hence $\vec{b} := \vec{OB} = [1, -1, -1]^T$
$C = (1, 1, 1)$
$D = (-1, -1, 1)$

Let φ be the angle AOB, which is the angle between $\vec{a}$ and $\vec{b}$. Then $\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \, \cos\varphi$. Specifically, $-1 = \sqrt{3}\,\sqrt{3}\,\cos\varphi$, hence $\varphi = \arccos(-1/3) \approx 1.9106\text{rad} \approx 109.47°$.

Hwk #4:

(a) By copycating the proof for the Cauchy-Schwarz inequality for vectors in $\mathbb{R}^n$, proof that for any two continuous functions f, g on the interval $[a, b]$, the inequality

$$\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\int_a^b f(x)^2\,dx}\,\sqrt{\int_a^b g(x)^2\,dx}$$

is true. (This inequality is called the Cauchy-Schwarz inequality for functions. It should be viewed as analogous to the CS inequality for vectors: $|\vec{f}\cdot\vec{g}| \le \|\vec{f}\|\,\|\vec{g}\|$)

(b) To begin appreciating the benefit of the inequality, find out what it tells you specifically about the integral $\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx$, an integral you will not be able to do by means of anti-derivatives. Use the CS inequality first for $f(x) = \sin x$ and $g(x) = \frac{1}{x}$, then for $f(x) = \sqrt{\frac{\sin x}{x}}$, $g(x) = \sqrt{x\,\sin x}$ to get a 2-sided estimate for the 'difficult' integral. (This technique doesn't always give numerically as good estimates as in this particular example.)

Note: The analogy between vectors and functions that is exploited here is studied more generally and systematically in a linear algebra course under the headings 'Abstract real vector spaces' and 'Inner product spaces'

Solution: (a) If g is the constant 0 on $[a, b]$, the inequality is obviously true. We now assume that g is not identically 0, and therefore $\int_a^b < g(x)^2\,dx > 0$. Consider the quantity $\int_a^b \left(f(x) + tg(x)\right)^2\,dx$, which is ≥ 0 for any real number t. We will later choose t conveniently. Expanding the square, we obtain

$$0 \le \int_a^b f(x)^2\,dx + 2t\int_a^b f(x)g(x)\,dx + t^2\int_a^b g(x)^2\,dx$$

If we now let $t := -\int_a^b f(x)g(x)\,dx / \int_a^b g(x)^2\,dx$ (which is possible since $\int_a^b g(x)^2\,dx > 0$), we obtain

$$0 \le \int_a^b f(x)^2\,dx - \frac{(\int_a^b f(x)g(x)\,dx)^2}{\int_a^b g(x)^2\,dx}$$

Clearing the denominator, moving the negative term over, and taking the square root proves the claim.

(b) For $f(x) = \sin x$ and $g(x) = \frac{1}{x}$, we obtain

$$\left|\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx\right| \le \sqrt{\int_{\pi/2}^{\pi} \sin^2 x\,dx}\,\sqrt{\int_{\pi/2}^{\pi} \frac{dx}{x^2}}$$

With

$$\int_{\pi/2}^{\pi} \sin^2 x\,dx = \frac{1}{2}\left[x - \sin x\,\cos x\right]_{\pi/2}^{\pi} = \frac{\pi}{4}$$

and

$$\int_{\pi/2}^{\pi} \frac{dx}{x^2} = \frac{1}{\pi}$$

we conclude $\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx \le \frac{1}{2}$.

Next, with $f(x) = \sqrt{\frac{\sin x}{x}}$ and $g(x) = \sqrt{x\,\sin x}$, we conclude

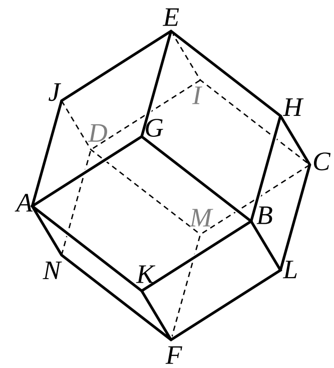
$$\left|\int_{\pi/2}^{\pi} \sin x\,dx\right| \le \sqrt{\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx}\,\sqrt{\int_{\pi/2}^{\pi} x\sin x\,dx}$$

Using $\int_{\pi/2}^{\pi} x\sin x\,dx = \pi - 1$, and $\int_{\pi/2}^{\pi}\sin x\,dx = 1$, we conclude after squaring that $\int_{\pi/2}^{\pi}\frac{\sin x}{x}\,dx \geq 1/(\pi-1)$.

Conclusion: $0.4669 \leq \int_{\pi/2}^{\pi}\frac{\sin x}{x}\,dx \leq 0.5$.

Hwk #5:

A rhombus is a quadrangle whose sides are all of the same length. A rhombododekahedron is a polyhedron with 12 faces, all of which are congruent rhombi. At each vertex, either four rhombi meet with their acute angles, or else, three rhombi meet with their obtuse angles. See the figure.
This does not work with an arbitrary rhombus. If you make the acute angle smaller (thus making the obtuse angle larger), the 'crown' above the zigzag $AGBHCIDJA$ becomes skinnier and taller, and the obtuse angle of the rhombus will become too large to fit below the crown as angle AGB.



Your job is to find the correct angle for a rhombus that is fit to build a rhombododekahedron. (No calculus here; just training your vector geometry and spatial vision a bit more.)

Hint: First choose a convenient cartesian coordinate system. Then set up equations describing that certain lengths are equal, to get coordinates of other points needed, then you can calculate the desired angle using the dot product.

Solution: Put the origin in the center of the square $ABCD$ and the x axis parallel to AB, the y axis parallel to BC, and the z axis passing through E and F. With a convenient choice of the unit length, we have $A = (-1,-1,0)$, $B = (1,-1,0)$, $C = (1,1,0)$, $D = (-1,1,0)$.

Then $E = (0,0,e)$ and $F = (0,0,-e)$ with e yet to be determined. (If you see that $EBFD$ should be a square as well, you know e without calculation, but I proceed to explain it assuming you don't see this.) The congruence of the rhombi requires that $\vec{EB}$ has the same length as $\vec{AB}$, namely 2. So $1^2 + 1^2 + e^2 = 2^2 + 0^2 + 0^2$. Hence $e = \sqrt{2}$.

Next $G = (0,-1,g)$ with g yet to be determined. Since the length of $\vec{AG}$ must be the same as the length of $\vec{EG}$, we conclude $1^2 + 0^2 + g^2 = 0^2 + 1^2 + (g-\sqrt{2})^2$, and therefore $g = \sqrt{2}/2$.

Now for the angle φ between the vectors $\vec{AG}$ and $\vec{AK}$, we have

$$\cos\varphi = \vec{AG}\cdot\vec{AK}/\|\vec{AG}\|\,\|\vec{AK}\| = \begin{bmatrix} 1 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix} /(1^2 + 0^2 + (\sqrt{2}/2)^2) = \frac{1/2}{3/2} = \frac{1}{3}\,.$$

Solution 2: A different solution, choosing another coordinate system: Let's put the origin of the coordiante system again in the center of the square $ABCD$, but this time let's have the axes diagonally: the x-axis goes through DB, the y-axis through AC. So, with a convenient unit length (different than the one before) we have $A = (0,-1,0)$, $C = (0,1,0)$ and $D = (-1,0,0)$, $B = (1,0,0)$. Similarly, since the z axis would now be FE, we have $F = (0,0,-1)$, $E = (0,0,1)$. With G being verticaly above the midpoint of AB, its coordinates are $G = (\frac{1}{2},-\frac{1}{2},g)$. For the length of $\vec{AG}$ to equal the length of $\vec{EG}$, we need $(\frac{1}{2})^2 + (\frac{1}{2})^2 + g^2 = (\frac{1}{2})^2 + (-\frac{1}{2})^2 + (1-g)^2$. This implies that $g = \frac{1}{2}$. Of course

$K = (\frac{1}{2}, -\frac{1}{2}, -g)$. Again, we can get for the angle φ between the vectors $\vec{AG}$ and $\vec{AK}$ that

$$\cos\varphi = \vec{AG} \cdot \vec{AK} / \|\vec{AG}\| \, \|\vec{AK}\| = \begin{bmatrix} 1/2 \\ 1/2 \\ g \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ -g \end{bmatrix} / \Big((\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + g^2 \Big) = (\tfrac{1}{2} - g^2)/(\tfrac{1}{2} + g^2) = \tfrac{1}{4}/\tfrac{3}{4} = \tfrac{1}{3}$$