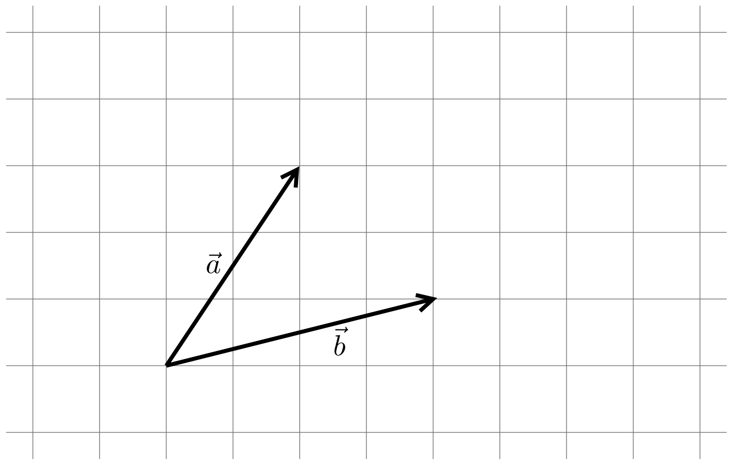
Homework

Math 247 – Honors Calculus 3

Fall 2015 – Jochen Denzler

Hwk #1:

The geometric vectors $\vec{a}$ and $\vec{b}$ are as given in the figure below. The coordinate lines drawn are at unit distances apart. The vector $\vec{c}$ is given in coordinates: $\vec{c} = [-1\,,\ 2]^T$.



(a) Draw $\vec{a} - \vec{b}$ and $\frac{1}{2}(\vec{a} + \vec{b})$ into the figure.
(b) Draw $\vec{c} = [-1\,,\ 2]^T$ into the same figure.
(c) Find the coordinates of $\vec{a}$ and $\vec{b}$, and calculate their dot product.
(d) Find the angle between $\vec{a}$ and $\vec{b}$ to a numerical precision of $1/100^{\text{th}}$ of a degree. Since you cannot read off the coordinates from a picture to sufficient precision, I'll tell you that the coordinates of $\vec{a}$ and $\vec{b}$ are indeed intended to be precise integers.

Hwk #2:

(a) Show that $2\|\vec{a}\|^2 + 2\|\vec{b}\|^2 = \|\vec{a} - \vec{b}\|^2 + \|\vec{a} + \vec{b}\|^2$. Draw a figure for illustration, and see why this formula is called the parallelogram identity.

(b) The dot product can be reconstructed from the norm: Indeed, show that $\vec{x} \cdot \vec{y} = \frac{1}{4}(\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2)$.

Hwk #3:

Let A, B, C, D be the vertices of a regular tetrahedron (a polyhedron whose four faces are congruent equilateral triangles). Let O be the center of this tetrahedron. Find the angle AOB. Interpretation: Chemists are interested in this angle b/c methane has a carbon atom in the center and a hydrogen atom at each of the vertices of the tetrahedron.

Hint: Take a cube, whose center is the origin, with the axes of a coordinate system parallel to the sides of the cube; choose four of its eight vertices in a checkerboard manner: if one vertex of the cube is chosen, then the immediately adjacent ones are not and vice versa. The chosen vertices form the corners of a tetrahedron. Draw a figure. It is easy to calculate (eg) the dot product $\vec{OA} \cdot \vec{OB}$ in terms of coordinates. The formula for this same dot product in terms of norms and angles can then be used to find the angle.

Hwk #4:

(a) By copycating the proof for the Cauchy-Schwarz inequality for vectors in $\mathbb{R}^n$, proof that for any two continuous functions f, g on the interval $[a, b]$, the inequality

$$\left|\int_a^b f(x)g(x)dx\right| \leq \sqrt{\int_a^b f(x)^2\,dx}\;\sqrt{\int_a^b g(x)^2\,dx}$$

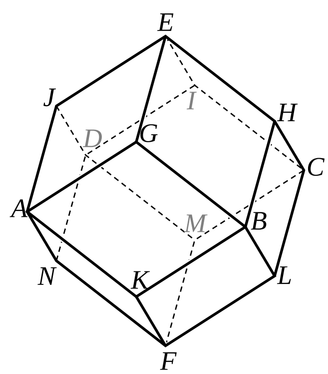
is true. (This inequality is called the Cauchy-Schwarz inequality for functions. It should be viewed as analogous to the CS inequality for vectors: $|\vec{f}\cdot\vec{g}| \leq \|\vec{f}\|\,\|\vec{g}\|$)

(b) To begin appreciating the benefit of the inequality, find out what it tells you specifically about the integral $\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx$, an integral you will not be able to do by means of antiderivatives. Use the CS inequality first for $f(x) = \sin x$ and $g(x) = \frac{1}{x}$, then for $f(x) = \sqrt{\frac{\sin x}{x}}$, $g(x) = \sqrt{x\,\sin x}$ to get a 2-sided estimate for the 'difficult' integral. (This technique doesn't always give numerically as good estimates as in this particular example.)

Note: The analogy between vectors and functions that is exploited here is studied more generally and systematically in a linear algebra course under the headings 'Abstract real vector spaces' and 'Inner product spaces'

Hwk #5:

A rhombus is a quadrangle whose sides are all of the same length. A rhombododekahedron is a polyhedron with 12 faces, all of which are congruent rhombi. At each vertex, either four rhombi meet with their acute angles, or else, three rhombi meet with their obtuse angles. See the figure.
This does not work with an arbitrary rhombus. If you make the acute angle smaller (thus making the obtuse angle larger), the 'crown' above the zigzag $AGBHCIDJA$ becomes skinnier and taller, and the obtuse angle of the rhombus will become too large to fit below the crown as angle AGB.

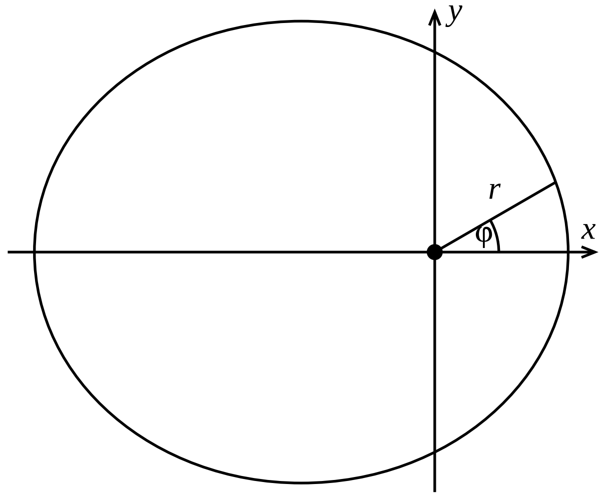


Your job is to find the correct angle for a rhombus that is fit to build a rhombododekahedron. (No calculus here; just training your vector geometry and spatial vision a bit more.)

Hint: First choose a convenient cartesian coordinate system. Then set up equations describing that certain lengths are equal, to get coordinates of other points needed, then you can calculate the desired angle using the dot product.

Hwk #6:

In astronomy, it is convenient to describe an ellipse (like, e.g., the orbit of the earth around the sun) in polar coordinates (with the sun at the origin, and the positive x axis through the perihelion (the point on the orbit of the earth that is closest to the sun). See figure (not to scale).



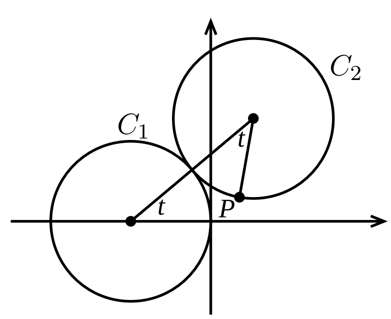
In polar coordinates, the orbit is given by $r = r_0/(1 + \varepsilon \cos\varphi)$, where $\varepsilon \in \,]0, 1[$ is called the eccentricity (a measure how much the ellipse deviates from circular shape). Obtain the equation in cartesian coordinates (x, y). The answer should be in the form (you fill in the '?').

$$\frac{(x-?)^2}{?^2} + \frac{y^2}{?^2} = 1$$

Hwk #7:

The cardioid is most easily described in terms of polar coordinates: $r = 2(1 - \cos\varphi)$. Its name comes from the Greek word for 'heart', and you'll see why when you graph this curve. So graph it carefully, but don't just steal the graph from a Valentine's card, that would be too corn(er)y!

A circle C_1 of radius 1 sits stationary with center $(-1, 0)$. Another circle C_2 of radius 1 touches it from the right, in the origin. This circle is soon to roll along the fixed circle C_1 without sliding. A point P is marked on the circle C_2. Initially it is the point where both circles touch. As C_2 rolls along C_1, the point P traces out a curve in the stationary plane. Use t for the angle (measured on C_1) of the point of contact of both circles. Give a vector valued function $t \mapsto \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ that describes the position of P as a function of t.



Show that the curve traced out by P is a cardioid.

Hwk #8:

Given a curve described in parametric form $t \mapsto \vec{x}(t)$, in the plane or in space, we may pretend that the parameter t represents a time and that $\vec{x}(t)$ is the position vector at 'time' t. Then the velocity is $\vec{x}'(t)$, and the speed is $\|\vec{x}'(t)\|$. The length of the curve between parameters t_0 and t_1 is $\int_{t_0}^{t_1} \|\vec{x}'(t)\|\,dt$.

Using this insight, calculate the length (perimeter) of the cardioid.

Hwk #9:

I love to collect airline miles. Let's assume I fly from Atlanta (ATL) to Frankfurt, Germany (FRA). How many miles is the shortest distance? I look up the following info on a map: ATL is at 84.4° western longitude and 33.65° northern latitude. FRA is at 8.6° eastern longitude and 50.1° northern latitude. The radius of the earth is 3975 mi.

Transforming from spherical to cartesian coordinates (the equator plane is the xy plane, with the Greenwich meridian going through the x axis); and using the dot product again, I can calculate the number of miles for this trip (along the shortest route, which is the arc of a circle centered at the center of the earth and connecting from ATL to FRA).

(Note: It is of course more expedient to use symbols $\lambda_{1,2}$ and $\varphi_{1,2}$ for the coordinates first, and postpone plugging in numbers until the end.)

Hwk #10:

Calculate the curvature of the helix (spiral staircase) given by $\vec{x}(t) = [r\cos t\,,\ r\sin t\,,\ ht]^T$.

Hwk #11:

Let $\vec{u} = [2,1,3]^T$, $\vec{v} = [-1,0,4]^T$, $\vec{w} = [2,-1,-3]^T$. Calculate $\vec{v} \times \vec{w}$, $\vec{u} \times (\vec{v} \times \vec{w})$, $\vec{u} \times \vec{v}$, $(\vec{u} \times \vec{v}) \times \vec{w}$.

With these same vectors from the previous problem, calculate $\vec{u} \cdot (\vec{v} \times \vec{w})$ and $\vec{w} \cdot (\vec{u} \times \vec{v})$.

Hwk #12:

Find the area of the triangle whose vertices are the points $A(1,1,3)$, $B(-2,3,0)$, $C(1,1,-2)$.

Hwk #13:

Given the vectors $\vec{u} = [u_1,u_2,u_3]^T$ and $\vec{v} = [v_1,v_2,v_3]^T$ in space, I have defined their cross product $\vec{u} \times \vec{v}$ to be the vector $\vec{w} = [u_2v_3 - u_3v_2\,,\ u_3v_1 - u_1v_3\,,\ u_1v_2 - u_2v_1]^T$. Show that indeed, $\|\vec{w}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2(1-\cos^2\varphi)$, where φ is the angle between $\vec{u}$ and $\vec{v}$.

Hwk #14:

Draw level curves of the function f given by $f(x, y) := |x| + |y|$, and describe the graph $z = f(x, y)$.

Same question for $g(x, y) = \sqrt{x^2 + y^2}$.

Hwk #15:

Try to understand the level sets of the function f given by $f(x, y) = 2x^2 - x^4 - y^2$. In particular use single-variable calculus and simple algebraic reasoning to find maxima of f. Make sure to sketch at least five level sets $\{(x, y) \mid f(x, y) = c\}$. Namely, for $c \in \{1, \frac{1}{2}, 0, -1, -4\}$. You may also find it useful to sketch graphs of a few single variable functions $g(x) := f(x, y_0)$ for various y_0. Try to avoid using technology that does 'multi-variable graphs', but feel free to enlist the help of technology for single variable graphs if this helps. The skill you are to train here is to piece single variable info together to get a multi-variable picture.

Hwk #16:

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \frac{xy}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. By using polar cordinates, draw the level curves of this function. As explained in class, this function is *not* continuous at $(0, 0)$. Show that nevertheless, all the single variable functions g and h given by $g(x) := f(x, y)$ for any choice of y, and $h(y) := f(x, y)$ for any choice of x, are continuous. For any fixed k, find the limit $\lim_{x\to 0} f(x, kx)$.

Hwk #17:

Cranking the previous example up a notch, consider the function f given by $f(x, y) := \frac{x^2y^4}{x^4+y^8}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

Show that each of the radial limits $\lim_{x\to 0} f(x, kx)$ and $\lim_{y\to 0} f(0, y)$ equal $0 = f(0, 0)$, but that, nevertheless, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist (and hence f is not continuous in the origin). Draw level lines of f to get insight into this function.

Can you describe a 'curve of approach' in the (x, y) plane along which the single variable limit exists, but is different from 0? That is, can you find $x(t)$ and $y(t)$ such that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$, but $\lim_{t\to\infty} f(x(t), y(t)) \neq 0$.

Can you also describe a curve of approach for which the limit does not exist?

Hwk #18:

Does $\lim_{(x,y)\to(0,0)} y \sin \frac{1}{x}$ exist, and if so, what is its value? Explain.

Hwk #19:

Consider the functions f and g given by $f(x,y) := x^2y + e^{xy} + y^3$ and $g(x,y) := \arctan\frac{y}{x}$. Calculate the following:

$$(a)\quad \frac{\partial f(x,y)}{\partial x}\,, \qquad (b)\quad \frac{\partial f(x,y)}{\partial y}\,, \qquad (c)\quad \frac{\partial g(x,y)}{\partial x}\,, \qquad (d)\quad \frac{\partial g(x,y)}{\partial y}\,,$$

Also calculate the following:

$$(a')\quad \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right), \quad (b')\quad \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right), \quad (c')\quad \frac{\partial}{\partial y}\left(\frac{\partial g(x,y)}{\partial x}\right), \quad (d')\quad \frac{\partial}{\partial x}\left(\frac{\partial g(x,y)}{\partial y}\right)$$

Compare (a') with (b') and (c') with (d').

Hwk #20:

More about the ellipse: Given the points $F_\pm = (\pm e, 0)$ in the plane (where e is some positive real number, not to be confused with the Euler number $2.718\ldots$), and a number $a > e$. Show that the set of those points $P = (x,y)$ in the plane that satisfy the condition $\|\vec{PF_+}\| + \|\vec{PF_-}\| = 2a$ is an ellipse $x^2/a^2 + y^2/b^2 = 1$. How does b relate to e and a? What is the eccentricity ε of the ellipse?

Hwk #21:

Reconsider the function f from Problem #8: $f(x,y) := \frac{x^2y^4}{x^4+y^8}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Write it in polar coordinates: $g(r,\varphi) := f(r\cos\varphi, r\sin\varphi)$. The partial derivative $\partial g(r,\varphi)/\partial r$ at $r = 0$ is a directional derivative of f (at the origin). Show that all directional derivatives at the origin vanish, so the graph has a horizontal tangent in each direction. Nevertheless, f is not even continuous at the origin.

Plot, for some choice of fixed φ (other than an integer multiple of $\pi/2$) the graph of the single variable function $g(\cdot,\varphi) : r \mapsto g(r,\varphi)$. Include information about the precise location of the maximum of this function.

Hwk #22:

Sketch level lines for the function $f(x,y) := x^3 - 3xy^2$. Choose levels 4, 1, 0, -1, -4. The most convenient way to do this is to use polar coordinates again. Look for a trig formula involving multiple angles that fits the situation (you'd likely not have memorized this formula to recognize it at first sight, that's why I say you should look for it).

This function is hand-picked to display a rare pattern in the level lines picture

Describe the graph of f in topographer's terms: where are the hills and the valleys? The point $(0,0)$ is said to feature a *monkey saddle* of this function f.

Hwk #23:

Find $Df(x,y)$ for $f(x,y) = x^y$, where $x > 0$.

Hwk #24:

Consider the following vector valued multi-variable functions:

$$f : (r,\varphi) \mapsto \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$$

defined for $\{(r,\varphi) \mid r > 0\,,\ \varphi \in \mathbb{R}\}$; and

$$g : (r,\vartheta,\phi) \mapsto \begin{bmatrix} r\sin\vartheta\,\cos\phi \\ r\sin\vartheta\,\sin\phi \\ r\cos\vartheta \end{bmatrix}$$

defined for $r > 0$, $\vartheta \in\,]0,\pi[$, $\phi \in \mathbb{R}$.

Find $Df(r,\varphi)$ and $Dg(r,\vartheta,\phi)$ by calculating the necessary partial derivatives and observing they are continuous.

Hwk #25:

Show that for $f(\vec{x}) := \|\vec{x}\|$ we have $\nabla f(\vec{x}) = \vec{x}/\|\vec{x}\|$ at $\vec{x} \neq \vec{0}$.

Hwk #26:

Consider the vector $\vec{e}$ and any (non-empty) level set of the function $g(\vec{x}) := \|\vec{x}-\vec{e}\|+\|\vec{x}+\vec{e}\|$. We know from #13 that the level set is an ellipse. In this problem we show the reflection property of the ellipse: A ray emanating from one focus $\vec{e}$ and reflected in the ellipse will pass through the other focus $-\vec{e}$. The reflection law in physics says that the incoming ray has the same angle with the normal to a curve as the reflected ray.

Prove the reflection property of the ellipse by checking the angle between the normal to the ellipse and the ray or reflected ray respectively. You can do everything without coordinates or components, just using vector notation.

Hwk #27:

The multi-variable chain rule says: $D(f\circ g)(\vec{p}) = Df(g(\vec{p}))Dg(\vec{p})$. Here is one specific example for which I ask you to caclulate all quantities involved in this equation and check the equality, all by explicit calculation.

$$f(x,y,z) := xyz^2 + (y-x)/(1+z^2), \quad g(\vartheta,\phi) = \begin{bmatrix} \sin\vartheta\,\cos\phi \\ \sin\vartheta\,\sin\phi \\ \cos\vartheta \end{bmatrix}.$$

(Motivation for this example: think of $f\circ g$ as a function on the unit sphere.)

Hwk #28:

Suppose that a piece of a level curve $g(x, y) = c$, near some point (x_0, y_0), where c is some constant, can be written as the graph of a function f: $y = f(x)$. Express the slope $f'(x_0)$ of this graph in terms of partial derivatives of g. Write your result both in mathematical notation and in physicists' notation with differential quotients (with z for the output variable of g and dy/dx expressed in terms of $\partial z/\partial x$ and $\partial z/\partial y$). *If you think a certain minus sign in your result looks weird, you're right: it does look weird, but it's still correct! Or rather I hope so, I haven't seen your solution; all I say is that the correct solution may have some weird looking detail.*

Hwk #29:

(For warmup only: this is predominantly single variable calculus)
Consider $f(x) := \int_0^\infty e^{-xt} \frac{\sin t}{t}\, dt$. You will not find an immediate antiderivative by which to evaluate this integral. Nevertheless, calculate $f'(x)$. [The Math 447 expert tells you that moving the x-derivative under the t-integral is legitimate here.] You should be able to evaluate the integral that you obtain for $f'(x)$.

What do you think $\lim_{x\to\infty} f(x)$ is? [The Math 447 expert tells you that in this example it is legitimate to move $\lim_{x\to\infty}$ past the integral sign.]

Finally, knowing $f'(x)$, $\lim_{x\to\infty} f(x)$ and the fundamental theorem of calculus, find $f(x)$. Specifically, what is $\int_0^\infty \frac{\sin t}{t}\, dt$?

The late physics Nobel prize winner Richard Feynman reports in his memoirs how, as a student, he got the reputation of being an integration wizard, because he was familiar with this 'differentiation under the integral sign technique', which his peers hadn't learned.

Hwk #30:

Use the multi-variable chain rule to determine $f'(x)$, when $f(x) := \int_0^x \frac{\sin(xt)}{t}\, dt$.

Analogous question for $g(x) := \int_{x/2}^{2x} \frac{e^{xt}}{t}\, dt$.

Again, we rely on the Math 447 expert, who tells us that it is legitimate to move derivatives past the integral sign in this example.

Hwk #31:

A quantity w depends on the coordinates x, y, z in 3-space as follows: $w = x^2 + y^2 + xyz$ (1). We study w especially on the plane given by $z = x + 2y$. Then we have there $w = x^2 + y^2 + xy(x + 2y) = x^2 + y^2 + x^2y + 2xy^2$ (2).

Now we calculate $\frac{\partial w}{\partial x}$ from (1): $\frac{\partial w}{\partial x} = 2x + yz$. On the plane, this simplifies to $\frac{\partial w}{\partial x} = 2x + y(x + 2y) = 2x + xy + 2y^2$.

Calculating $\frac{\partial w}{\partial x}$ on the plane directly from (2), we get $\frac{\partial w}{\partial x} = 2x+y(x+2y) = 2x+2xy+2y^2$. We clearly have a discrepancy by a term xy. What is wrong? Clear up the confusion. (This requires some text as well as formulas.)

Hwk #32:

This example is taken from *I. Rosenholtz, L. Smylie: "The only Critical Point in Town" Test*, Mathematics Magazine **58**(1985), 149–150.

Show that the function

$$g : (x, y) \mapsto y^2 + 3(y + e^x - 1)^2 + 2(y + e^x - 1)^3 \, , \; \mathbb{R}^2 \to \mathbb{R}$$

has exactly one critical point, and that this point is a relative mimimum.

Furthermore explain why this point is NOT an absolute minimum.

Hwk #33:

This example is taken from Marsden-Tromba: Show that the function f given by $f(x, y) = (y - 3x^2)(y - x^2)$ has a critical point in the origin, which is neither a relative minimum nor a relative maximum. What kind of '***'definite is the Hessian?

Show also that all single-variable radial functions $t \mapsto f(t\cos\phi, t\sin\phi)$ have a relative minimum at $t = 0$.

Hwk #34:

This example is geometrically appealing, but alas calculationally lengthy. This is why I give you the intermediate steps and hints to navigate you through. Ideally, it should be done with the help of symbolic algebra software, and you are welcome to use this tool, if available.

We want to find a shortest connection between two plane curves, namely $y = x^2 + 2$ and $y = \frac{1}{2}(x-1)^2$. A precise plot is attached. Choose points $P = (a, a^2+2)$ on the first parabola and $Q = (b, \frac{1}{2}(b-1)^2)$ on the second and minimize the square of the distance. Determine all critical points and classify them. Does one of them probide a *global* minimum? Why?

Hint 1: While it is possible to take one of the equations for a critical point and solve it for b via by means of the quadratic formula, and then plug in the result in the other equation, this is tedious. It is more straightforward to take successively linear combinations of the two equations with the strategy of first eliminating b^3, then b^2, then b, until one polynomial equation in a remains.

Hint 2: After an obvious factorization of this polynomial equation, an easy solution $a = 1$ can be guessed, and when this is factored off, a 4th order polynomial remains that can be factored into two quadratics with integer coefficients; indeed one factor is $a^2 + 2a + 3$.

Hwk #35:

Suppose in the following matrices, the starred entries are not known. Which of the five possibilities 'positive definite', 'positive semidefinite (but not definite)', 'negative definite', 'negative semidefinite (but not definite)', 'indefinite' remains a possibility, based on knowledge only of the known entries?

$$(a) \begin{bmatrix} 3 & * \\ * & * \end{bmatrix} \qquad (b) \begin{bmatrix} * & * \\ * & -5 \end{bmatrix} \qquad (c) \begin{bmatrix} * & 6 \\ 6 & * \end{bmatrix} \qquad (d) \begin{bmatrix} 3 & * \\ * & -1 \end{bmatrix} \qquad (e) \begin{bmatrix} 0 & 1 & * \\ 1 & * & * \\ * & * & * \end{bmatrix}$$

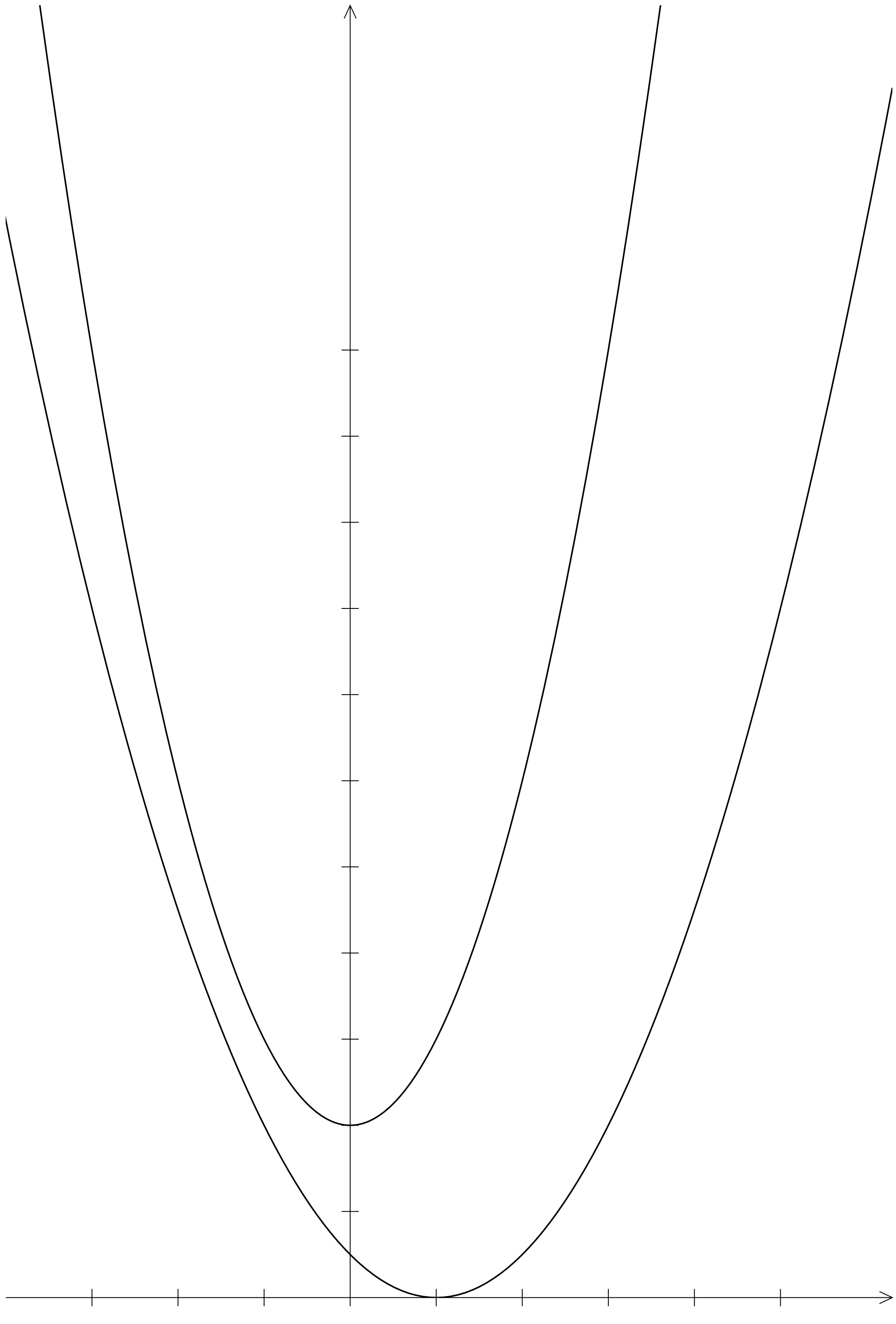


Figure 1: Figure for Hwk # 34

Consider the following: While the Hurwitz test was worded in a way to calculate determinants starting from the top, the order in which the variables are listed (and thus determine entries of the matrix) is not essential for definiteness of a matrix; so you could use the determinants in the Hurwitz test starting at any diagonal element and then calculating 2×2, 3×3, etc. determinants, adding any one variable (row and column) at a time.

Hwk #36:

Find the absolute minimum and absolute maximum of $x^2+(y-1)^2+z^2-xyz$ on the ball $x^2+y^2+z^2\le 3^2$. *Hint: For the boundary consideration, use the xz plane as equator plane for the spherical coordinates, to benefit from the symmetry of the problem. Otherwise formulas get obnoxiously messy.*

Hwk #37:

Redo the boundary part of the calculations from the previous problem using Lagrange multipliers.

Hwk #38:

This problem gives you the most celebrated use of Lagrange multipliers, but it requires some intrduction to appreciate it. (The calculations aren't bad at all.)

A famous task in linear algebra and matrix theory is to find *eigenvalues* of a given matrix. If A is a square matrix and you can find a *non-zero* vector v such that Av is actually a multiple of v, i.e., λv where λ is a number, then we call λ an eigenvalue of the matrix A (and v an eigenvector). For instance, the matrix $A=\begin{bmatrix} 3 & 2\\ -3 & -4\end{bmatrix}$ has 2 as an eigenvalue and $\begin{bmatrix} 2\\ -1\end{bmatrix}$ as a corresponding eigenvector, because

$$\begin{bmatrix} 3 & 2\\ -3 & -4\end{bmatrix}\begin{bmatrix} 2\\ -1\end{bmatrix}=2\begin{bmatrix} 2\\ -1\end{bmatrix}$$

It also has -3 as an eigenvalue with $\begin{bmatrix} 1\\ -3\end{bmatrix}$ as an eigenvector. Of course multiples of eignevectors are again eigenvectors, e.g., if $Av=2v$ then also $A(7v)=2(7v)$. — In the example, there are only these two numbers $\lambda_1=2$ and $\lambda_2=-3$ that are eigenvalues. If you try to find $v=\begin{bmatrix} v_1\\ v_2\end{bmatrix}$ solving $Av=\lambda v$ for any other λ you will only get the solution $v_1=v_2=0$, i.e., only the zero vector. (Try it, just to gain familiarity with the notions.)

This problem is about eigenvalues of *symmetric* matrices. They play a role in studying definiteness of symmetric matrices. In physics, they are key concepts in describing rotating motions of rigid bodies. To every body, there is associated a symmetric 3×3 matrix called its 'tensor of inertia', whose eigenvectors point in the directions of such axes about which the body can rotate without wobbling (i.e., in a balanced way). The eigenvalues are called the moments of inertia about these axes.

To every symmetric $n\times n$ matrix A we associate the quadratic function $f(x) := x^TAx$ where $x=[x_1,\,\ldots,\,x_n]^T\in\mathbb{R}^n$. We try to minimize or maximize $f(x)$ under the constraint $x^Tx=1$ (i.e., for x on the unit sphere).

(a) Write out $f(x)$ in components x_i for a 3×3 matrix A whose entries are called a_{ij}. Explain why a global maximum and a lobal minimum of $f(x)$ on the sphere are a-priori guaranteed to exist.

(b) Use the Lagrange multiplier method to set up equations satisfied by the x providing a minimum or a maximum. (You may have written all these in components; but now make sure to rewrite the whole stuff again in matrix and vector form.) While you are not asked to actually solve for x (that would be very tedious, involving a cubic equation for λ), I ask you to express the value of f at the minimum and maximum in terms of the Lagrange multiplier. [Be aware that when finding the max vs the min, x and λ will typically refer to numerically different quantities in these two cases.]

You have just proved that every symmetric 3×3 matrix has (at least) two real eigenvalues. (Actually, if A is a multiple of the identity matrix, these two eigenvalues coincide.) And with jut a bit more writing, the same can be done for symmetric $n \times n$ matrices.

The method can be cranked up a bit, by throwing in further constraints, to prove that every symmetric $n \times n$ matrix has n real eigenvalues (some of which may coincide). This may well be among the most important pieces of insight in undergraduate mathematics, and it's a pity that it often falls between the cracks of separating Calc 3 and Linear Algebra into independent courses of the curriculum.

(c) Show, in a very brief calculation: If A is positive definite, then all its eigenvalues are positive. If A is positve semidefinte, then all of its eigenvalues are ≥ 0. FYI: The converse is also true; so indeed a symmetric matrix is positve definite (resp. emidefinite) IF AND ONLY IF all of its eigenvalues are positive (resp. non-negative). This statement is sponsored by the above proof (a), (b) and some extra dose of linear algebra. It is the launch pad for proving the Hurwitz and Gershgorin tests I gave you before.

Hwk #39:

Think of the task of finding the absolute maximum of $x^2 + \frac{1}{2}y^2 + y^4 - xy$ on the set S given by $(x-1)^2 + |y + y^3| \leq 5$. The purpose of this problem is NOT that you would actually do calculations to FIND the maximum (which would require numerical methods). Rather, in preparation for such a search. I want you to use the Hessian to conclude that the maximum exists and is on the BOUNDARY of the set S.

The message here is: While modest problems can already lead to prohibitively complicated calculations that may need numerical tools, simple analytic arguments may still be able significantly to reduce the amount of labor in a numerical search.

Hwk #40:

You may or may not have seen the following formula (called Heron's formula): The area of a triangle with sides a, b, c is $\sqrt{s(s-a)(s-b)(s-c)}$ where s is the semiperimeter $\frac{1}{2}(a+b+c)$. Show that among all triangles with a given perimeter $2s = a + b + c$, the area takes an absolute maximum exactly for the equilateral triangle. (Explain first why an absolute minimum exists before calclulating it.)

Hwk #41:

Let T be the set $\{(x, y) \mid 0 \leq x \leq \pi,\ 0 \leq y \leq x\}$. Draw a figure of this set. Then evaluate the integral $I := \int_T \sin x \, \sin y \, d(x, y)$ in two ways: as iterated integral in either order.

Hint: Make sure you get the limits of integration right. If any of your calculations leaves a dangling x or y in the result you sure haven't gotten the limits right. This alert applies to all MV integral problems.

Hwk #42:

Let A be the set $\{(x, y) \mid 1 \leq x^2+y^2 \leq 4,\ x, y \geq 0\}$. Draw a figure of this set. Then evaluate the integral $I := \int_A x^2 y \, d(x, y)$ in two ways: one version using cartesian coordinates, and one using polar coordinates.

Using cartesian coordinates here is a bit dumb, admittedly. But I am asking that you do it anyways, to see the comparison with polar coordinates, and as a training to deal with the limits of integration correctly. Note that one order of integration in cartesian coordinates is easier to calculate than the other. Can you see which, and why?

Hwk #43:

Physicists are familiar with the following phenomenon: If you let a massive ball and a massive cylinder roll down an incline, then the ball rolls more rapidly than the cylinder. The reason is that part of the potential energy gained when losing height is converted into kinetic energy for the forward movment, whereas another part is converted into 'internal' (rotational) kinetic energy, because the object is rolling rather than just sliding. This rotational energy is lost to the forward motion.

You may know the formula $\frac{1}{2}mv^2$ (half mass times velocity^2) for the translation energy. There is a similar formula $\frac{1}{2}I\omega^2$ for the rotation energy, where ω measures how many radians per time unit an object rotates. The quantity I is called 'moment of inertia' and it depends on the mass distribution in the body. Mass that is closer to the rotation axis counts less because it does not move as fast as mass that is farther away from the rotation axis.

The formula for I is: $I = \int_{\text{body}} s^2 \rho d\text{vol}(x, y, z)$. Here ρ is the density (which may depend on (x, y, z), but in this problem we assume it is constant). s denotes the distance from the rotation axis, which you have to express in terms of x, y, z or whatever coordinates you use.

Given this wisdom, I ask you to find I for a cylinder of radius R and height h, and also for a ball of radius R. In either case, these objects rotate about a symmetry axis. You are to express the result in the form: number times (total mass) times R^2. Remember that the total mass is volume times density ρ.

The larger the number in front of 'mass times R^2' is, the higher the proportion of energy that is used for the rotation.

Hwk #44:

Now we rotate a cube $-a \leq x, y, z \leq a$ about an axis through the origin. The axis goes in the direction of a vector $\vec{v}$.

First draw a generic picture of a vector $\vec{x} = [x,\ y,\ z]^T$ and a vector $\vec{v} = [v_1,\ v_2,\ v_3]^T$ (both starting at the origin) and find a formula for the distance s of the tip of $\vec{x}$ (i.e., of the point (x, y, z)) from the axis that goes along the vector $\vec{v}$.

Then calculate the moment of inertia for this rotation (expressed as number times mass times a^2). Surprise: The final result will not depend on $\vec{v}$ — (To those who know about the tensor of inertia and the role eigenvalues play there, this surprise will be expected; but these wise folks, that's not us, for the time of Calc 3.)

Hwk #45:

Find the center of mass of the 'full' cardioid, i.e., of the area enclosed by the curve $r = a(1-\cos\varphi)$ in polar cordinates. Is it the same as the center of mass of the *curve* $r = a(1-\cos\varphi)$, which we calculated to be $(-\frac{4}{5}a, 0)$?

Hwk #46:

Find the center of mass of the semicircle $\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ where t goes from $-\pi/2$ to $\pi/2$.

Hwk # 47:

Integrate the vector field $[xy,\ yz,\ xz]^T$ over the curve γ parametrized by $\vec{x}(t) = [t,\ t^2,\ t^3]^T$ for $0 \leq t \leq 1$. (Here of course, x, y, z are the components of the vector $\vec{x}$.)