— The Joy of Power Series; A Mathematical Symphony in n Movements —
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PS 0: Power Series: Overview

Power series owe their importance to two facts: Firstly, every "decent" function can be expressed in terms of a power series, and secondly, calculations with power series are nearly as easy as calculations with polynomials.

Here are some examples: they comprise an almost complete basic power series toolkit. You will already have seen most of them in the lecture, and the purpose of this manuscript is to give you a written, slightly expanded, version of the lecture.

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots = \sum_{n=0}^{\infty} x^n \qquad (|x|<1) \tag{1}$$

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+5x^4+\cdots = \sum_{n=0}^{\infty} (n+1)x^n \qquad (|x|<1) \tag{2}$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+-\cdots = \sum_{n=0}^{\infty} (-1)^n\frac{x^{2n}}{(2n)!} \qquad (\text{any } x) \tag{3}$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+-\cdots = \sum_{n=0}^{\infty} (-1)^n\frac{x^{2n+1}}{(2n+1)!} \qquad (\text{any } x) \tag{4}$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+-\cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} \qquad (-1<x\le 1) \tag{5}$$

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+-\cdots = \sum_{n=0}^{\infty} (-1)^n\frac{x^{2n+1}}{2n+1} \qquad (|x|\le 1) \tag{6}$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (\text{any } x) \tag{7}$$

$$\tan x = x+\frac{x^3}{3}+\frac{2}{15}x^5+\frac{17}{315}x^7+\ldots \qquad (|x|<\pi/2) \tag{8}$$

$$\sqrt{1-x} = 1-\frac{1}{2}x-\frac{1}{2{\cdot}4}x^2-\frac{1{\cdot}3}{2{\cdot}4{\cdot}6}x^3-\ldots-\frac{1{\cdot}3{\cdot}5\cdots(2n-3)}{2{\cdot}4{\cdot}6{\cdot}8\cdots(2n)}x^n-\ldots \qquad (|x|\le 1) \tag{9}$$

In (8), the formula for n^{th} coefficient would be quite unwieldy, so I have omitted it.

You know formula (1) already; it's the formula for a geometric series. The knowledge to express many functions as convergent series $\sum_{n=0}^{\infty} c_n x^n$, at least on some interval, may well be viewed as one of the very big highlights of calculus. If there existed a Mount Rushmore of Calculus, power series would find their place there next to derivative and integral. Of course the foundational notion underlying all three of these is the notion of a limit: it will therfore get George Washington's place on the Mount Rushmore of Calculus, and this gives us four faces `;-)`

PS 1: A Power Series that Appoints Itself Object of Study

It is a somewhat untypical, but instructive calculation that leads us from very simple facts to (3) and (4): We will assume $x>0$. The calculation for $x<0$ is similar. From $\cos t\le 1$, we conclude that $\int_0^x \cos t\,dt \le \int_0^x 1\,dt$, i.e., $\sin x\le x$. This procedure can be repeated, and you get

$\int_0^x \sin t\,dt \leq \int_0^x t\,dt$, which evaluates to $\cos x \geq 1 - x^2/2$ after a short calculation. The next step is $\int_0^x \cos t\,dt \geq \int_0^x (1 - t^2/2)\,dt$, or in other words $\sin x \geq x - x^3/6$. as seen in the homework

If you repeat the calculation over and over again, you construct, step by step, longer partial sums of the series in (3) and (34. But the calculation does not give any information yet, whether these series are actually convergent for any x. You can however use the ratio test to see that they are (absolutely) convergent for each x. Given that you come up with alternating signs, you may consider trying Leibniz' test to prove convergence. However, for large x, the sequence of terms (omitting the signs) will not be nondecreasing any more, so Leibniz' test doesn't apply. This problem could be fixed, but we wouldn't even bother trying, because Leibniz would *not* give us *absolute* convergence. So we are much better off with the ratio test.

Knowing now that the series (3) converges, i.e., that the sequence of partial sums converges, we want to know its limit of course. But given that these partial sums are alternatively $\leq \cos x$ and $\geq \cos x$, the limit couldn't be anything but $\cos x$ itself. This proves the equation (3) for all x.

The nice feature of this calculation is that, without inputting any theory about series at all, you have been led to an expression like the one on the right hand side of (3), a result that necessitates the study of series; and then, with the basic theory of series alone, but nothing yet specific to power series, you get equation (3), which is surprising enough in itself to warrant a closer study of power series. Hence the title of this section. Equation (4) is obtained in the very same way as (3), out of the same calculation.

PS 2: Definition and Convergence Properties of Power Series

We call any formal expression

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots = \sum_{n=0}^{\infty} c_n x^n$$

a power series centered at 0. Here, $c_0, c_1, c_2 \ldots$ are numbers. Similarly, we call

$$d_0 + d_1(x-a) + d_2(x-a)^2 + d_3(x-a)^3 + \ldots = \sum_{n=0}^{\infty} d_n(x-a)^n$$

a power series centered at a. We will first study for which x these series can converge. And next you will learn that in many respects, you can calculate with power series as if they were polynomials. The one, notable exception being that in $\sum_{n=0}^{\infty} d_n(x-a)^n$, don't even think of expanding the parentheses. Doing so may be legitimate sometimes (sometimes, it will not be legitimate), but even when it is legitimate, it will produce a horrible mess that will not be good for anything.

The basic result about convergence is that every power series converges in some interval (which may also be infinite or shrink to a point) and diverges outside that interval: If the power series is centered at a, the only possibilities are:
(i) absolute convergence for $x \in (a-R, a+R)$, divergence for $|x-a| > R$, where R is some positive number depending on the power series under investigation, or
(ii) absolute convergence for all x (this corresponds to $R = \infty$), or
(iii) convergence for $x = a$ (trivially: $d_0 + 0 + 0 + 0 + \cdots = d_0$) only, else divergence; (this corresponds to $R = 0$).

You do not need a separate formula to find R, because you will use either the ratio or the root test to determine for which x the series converges, and the result will automatically fall into one of the three categories given here, and you can just read off R. For any given power series, if it goes in case (i), a special discussion will be needed to determine whether the series converges (conditionally or absolutely) or diverges at each of the boundary points $a + R$ and $a - R$. This looks like bad news, but the good news is that you will always (a) either have a fair chance to decide this question using Leibniz' or comparison tests, or else (b) the question of convergence at these two single points wouldn't be important enough to work hard on it. The number R is called *radius of convergence*. Accept the name for the moment. To explain the reason for such a funny name (why "radius"?) requires to refer to complex numbers (PS 4 below).

Now here is what you may do with power series: Generally, the operations you probably wish to carry out are permitted for x in the interior of the interval of convergence. A power series may happen to converge at a boundary point of the interval of convergence. But except when explicitly specified otherwise, you will disregard this information and only claim your calculations to be valid for x in the interior of the interval of convergence, not claiming anything for x on the boundary. The one exception is section PS 3.3 below, which specifically addresses the boundary points.

PS 3: Calculating With Power Series

PS 3.0: Adding and Subtracting Power Series, etc

So straightforward that I didn't dare to number this one as 3.1. For instance: From

$$\tfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots \qquad \text{and} \qquad \tfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - + \ldots$$

you conclude by addition (term by term) that

$$\tfrac{1}{1-x} + \tfrac{1}{1+x} = 2 + 2x^2 + 2x^4 + \ldots$$

This is true for those x for which both input series are convergent, i.e., for $|x| < 1$. And multiplying series (1) by x would give $\frac{x}{1-x} = x + x^2 + x^3 + x^4 + x^5 \ldots$, distributing the factor x over all terms in the "sum of infinitely many terms". You could do these things to any convergent series, and of course you could also multiply (1) by $\ln x$ to get $\frac{\ln x}{1-x} = \ln x + x \ln x + x^2 \ln x + x^3 \ln x + x^4 \ln x + \ldots$. This latter is however not a power series and will therefore rarely be useful (even though it is legitimate for $0 < x < 1$). So if you ever feel the urge to multiply a power series by a function that is not a power, think again, if you really want it, and why.

PS 3.1: Term by Term Differentiation

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for (at least) $|x - a| < R$, then f is differentiable, and it holds $f'(x) = \sum_{n=0}^{\infty} c_n n (x-a)^{n-1}$, for these same x satisfying $|x - a| < R$.

As an example, you obtain formula (2) by taking the derivative of formula (1), and you obtain (3) as the derivative of (4). In short, differentiating power series is just like differentiating polynomials.

(If the series for f happens to converge at a boundary point of the interval of convergence as well, the same needn't be true any more for the derived series.)

PS 3.2: Term by Term Integration

Integrating power series is also just like integrating polynomials.

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for (at least) $|x-a| < R$, then $\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$ for these same x satisfying $|x-a| < R$.

As an example, you can use a variant of (1), namely

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n$$

and take the integral term by term to get formula (5); however, you get (5) only for $-1 < x < 1$ from this argument, but not for $x = 1$. Only PS 3.3 will yield the result for $x = 1$ then.

Similarly, you can start with $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + - \cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n}$, which is another variant of (1). By integrating with respect to x, you get (6) for $|x| < 1$. Its validity for $x = \pm 1$ as well is not a consequence of term by term integration, but is again due to PS 3.3.

3.3: Plugging in Boundary Points

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for (at least) $|x-a| < R$, and the series on the right hand side still happens to converge at a boundary point, say, at $x = a + R$, then its value is indeed $\lim_{x\to a+R-} f(x)$. *Or similarly, if the series happens to converge at $x = a - R$, then its value is indeed* $\lim_{x\to a-R+} f(x)$.

In practice, f will be given by a formula representing a continuous function, and therefore this statement reduces to the following, somewhat sloppy statement:
If continuous expression = power series for x in the interior of the interval of convergence and if the series still happens to converge at some boundary point of that interval, then the equation also holds at this boundary point. [This is sometimes called Abel's limit theorem.]

For example, you got (5) and (6) originally only for $|x| < 1$. But now you conclude that the equality continues to be true for $x = 1$, because the right hand side still converges for $x = 1$, in each of the two series.

This means that you get the surprising formulas: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots = \ln 2$, and $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \pi/4$. But don't use them for *numerical* calculation of π or $\ln 2$. That would be about the most tedious and inefficient way conceivable to do the job. Insight in the mentioned formulas is the main use of this rule.

In contrast, you would not have (5) for $x = -1$, because the right hand side of (5) diverges for $x = -1$. And clearly, you should also not use (1) with $x = -1$ to conclude

$$\frac{1}{2} \text{ “=” } 1 - 1 + 1 - 1 + 1 - + \cdots \qquad\qquad \text{NOT ok!}$$

because the right hand side does not converge. If you feel that such a phony statement should in some sense be considered as reasonable, you will be happy to learn that some great ancestors of calculus (like Euler) actually did such stuff. But writing this kind of formulas is really like smoking and filling the gasoline tank at the same time, and it is only because there were no gas stations around in Euler's times that the whole thing didn't blow up in Euler's face right away. And if this joke doesn't convince you, why don't you commit the outrage again and plug in (illegally) $x = -1$ into (2)? Do you still feel safe ground under your feet with what you would get? Or would you like what you would get by (illegally) plugging $x = 2$ into equation (1)? Probably (hopefully) not. Which is why we insist on convergence.

PS 3.4: Multiplying Power Series

If you have two convergent power series centered at the same *a, you are permitted to multiply them according to the distributive law, like polynomials. The resulting power series will have a radius of convergence at least as large as the smaller of the radii of convergence of the two factors.* In exceptional cases, the radius of convergence may be larger, but in most cases it will not be larger:

For instance, the series (2) arises from multiplying (1) with itself:

$$\begin{aligned}\frac{1}{(1-x)^2} &= (1+x+x^2+x^3+\cdots)(1+x+x^2+x^3+\cdots) = \\ &= 1{\cdot}1 + (1{\cdot}x + x{\cdot}1) + (1{\cdot}x^2 + x{\cdot}x + x^2{\cdot}1) + (1{\cdot}x^3 + x{\cdot}x^2 + x^2{\cdot}x + x^3{\cdot}1) + \cdots = \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\end{aligned}$$

Note the intermediate step, which shows how you organize such a calculation: you look first for products contributing a constant (there is only one such product), then for products contributing to x^1, and so on working towards higher powers of the resulting series. Similarly, you get

$$\begin{aligned}\frac{1}{(1-x)^3} &= (1+2x+3x^2+4x^3+5x^4+\cdots)(1+x+x^2+x^3+x^4+\cdots) = \\ &= 1{\cdot}1 + (1{\cdot}x + 2x{\cdot}1) + (1{\cdot}x^2 + 2x{\cdot}x + 3x^2{\cdot}1) + (1{\cdot}x^3 + 2x{\cdot}x^2 + 3x^2{\cdot}x + 4x^3{\cdot}1) + \cdots = \\ &= 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \cdots\end{aligned}$$

Since each of these series converges for $|x| < 1$, the product is guaranteed to converge (at least) for $|x| < 1$, too. Actually, it converges only for these x.

You may try to multiply $\sum_{n=0}^{\infty} x^n/\pi^n = (1-\frac{x}{\pi})^{-1}$ and $\sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)! = \sin x$. The result will be a power series whose sum is given by the expression $(\sin x)/(1-\frac{x}{\pi})$. The theorem assures you that the resulting series converges at least in the interval $|x| < \pi$, which is the interval of convergence of the first factor. (The second factor, namely the series for the sine, converges everywhere.)

If you actually calculate a few terms of the product series, you will see that
(a) the calculation is routine, but the result is not enlightening; the coefficients remain messy and don't simplify;
(b) you have no idea about the general formula for the n^{th} coefficient, except the messy one that will involve '$\ldots$';
(c) you would therefore not see how you could actually calculate the precise radius of convergence of that series, say by the ratio test; the trouble being that the expression a_{n+1}/a_n is already unwieldy.

To address these possibly bad surprises, be advised that
(a,b) it is often useful to have the first few terms from a routine calculation, even without a general term;
(c) so much the better that the theorem tells you an interval where the series converges *at least*. If the series chooses to converge in an even larger interval without our knowing this, that wouldn't harm any of our plans; at worst we would ignore some good news that remains hidden

from our knowledge;
(d) actually this particular series does converge in a larger interval, but to see this requires advanced knowledge. This is a peculiar example. If I had taken a different number instead of π, say $\frac{22}{7}$, the "convergence gratuity" would not have been present any more.

PS 3.5: Long Division of Power Series

Long division of power series is straightforward and practical, despite being omitted from most calculus textbooks. is so cute that it's a pity the book omits it. There are two caveats that have to be observed: (1) Unlike long division of polynomials (where you start with the highest power and work towards decreasing powers), with power series, you start with the constant term and work your way up, towards increasing powers. (You couldn't start from the top, because there is no top!)

Have a look at the example before we discuss the second caveat: The example shows you the calculation of the first few terms of a power series whose sum is $\tan x$, obtained by long division from the power series (4) for $\sin x$ and (3) for $\cos x$.

$$
\begin{array}{r|l}
\begin{array}{rl}
 & x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \dfrac{17}{315}x^7 \cdots \\
\hline
1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720} + - \cdots \Big) & x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040} + - \cdots \\
 & x - \dfrac{x^3}{2} + \dfrac{x^5}{24} - \dfrac{x^7}{720} + - \cdots \\
\hline
 & \phantom{x -{}} \dfrac{x^3}{3} - \dfrac{x^5}{30} + \dfrac{x^7}{840} \cdots \\
 & \phantom{x -{}} \dfrac{x^3}{3} - \dfrac{x^5}{6} + \dfrac{x^7}{72} \cdots \\
\hline
 & \phantom{x - \dfrac{x^3}{3}} \dfrac{2x^5}{15} - \dfrac{4x^7}{315} \cdots \\
 & \phantom{x - \dfrac{x^3}{3}} \dfrac{2x^5}{15} - \dfrac{x^7}{15} \cdots \\
\hline
 & \phantom{x - \dfrac{x^3}{3} - \dfrac{x^5}{3}} - \dfrac{17}{315}x^7 \cdots
\end{array}
&
\begin{array}{l}
\text{auxiliary calculations:} \\[1em]
6! = 720\ ,\quad 7! = 5040 \text{ etc} \\[1em]
-\frac{1}{6} + \frac{1}{2} = \frac{1}{3}\ ,\quad \frac{1}{120} - \frac{1}{24} = \frac{-1}{30} \\[1em]
\qquad -\frac{1}{5040} + \frac{1}{720} = \frac{1}{840} \\[3em]
-\frac{1}{30} + \frac{1}{6} = \frac{2}{15}\ ,\quad \frac{1}{840} - \frac{1}{72} = \frac{-4}{315}
\end{array}
\end{array}
$$

Clearly, this calculation is not appropriate to find a formula for the general (n^{th}) term of a series representing $\tan x$. However, you can get as many terms of the series as you like, and you are guaranteed that the resulting series converges in *some* interval.

And here is the other caveat: the resulting power series is guaranteed to converge in *some* small interval, but the size of this interval cannot be predicted so easily. It will often be smaller than the intervals of convergence of either numerator or denominator. You will now see why this is not weird at all, but actually a rather natural and expected thing to happen:

Indeed, even though you have no means (at 1st year level) to predict the *exact* radius of convergence, you could predict that the radius of convergence could *not* be *larger than* $\pi/2$, in spite of the fact that the series for $\sin x$ and $\cos x$ converge everywhere. Why? *Think a while, before you look up the answer in the footnote.*[1] As a matter of fact, the radius of convergence is exactly $\pi/2$, and this is junior or senior level knowledge.

An Application:

Here we see why the exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ equals e^x.

First we note that the series converges for all x (using the ratio test). We'll temporarily call its yet unknown sum $f(x)$. Term-by-term differentiation shows us that $f'(x)$ is the very same series $1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \ldots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$. So $f'(x) = f(x)$.

Since $f(x)$ is its own derivative, we may suspect that $f(x) = e^x$; but note that at least all multiples of e^x are also their own derivatives. Are there further functions (other than $c\,e^x$) that we have not been aware of that are also their own derivatives? It turns out there are not:

Indeed if $f'(x) = f(x)$, then

$$\frac{d}{dx}(f(x)e^{-x}) = f'(x)e^{-x} - f(x)e^{-x} = 0\ .$$

So $f(x)e^{-x}$ must be constant; i.e., we have already shown that $f(x) = c\,e^x$.

From the series, we also see that $f(0) = 1$. This tells us that the constant is 1. We conclude that $f(x) = e^x$.

PS 3.5: Composition of Power Series

Suppose $f(y) = c_0 + c_1 y + c_2 y^2 + \ldots = \sum_{n=0}^{\infty} c_n y^n$ and we substitute $y = g(x) = d_1 x + d_2 x^2 + d_3 x^3 + \ldots = \sum_{n=1}^{\infty} d_n x^n$. The key assumption is here that $g(0) = 0$, i.e., $d_0 = 0$.

Then it is allowed to substitute the power series $d_1 x + d_2 x^2 + d_3 x^3 + \ldots$ for y into the power series $c_0 + c_1 y + c_2 y^2 + \ldots$ to get a power series for $f(g(x))$. If the power sereis for f and g have positive radius of convergence, the the resulting power series has also positive radius of convergence. But we have no means of predicting the precise radius of convergence.

— OUTLOOK: —

The only series in our list that has not been explained so far is (9). Its study is best postponed until we deal with Taylor's formula. The main program ahead of us will be: Given any "decent" function, can we find a power series that converges to this function, at least on some interval? The answer will be "yes" in most cases, and this is gorgeous! It means roughly "every decent function may be viewed as something like a polynomial of possibly infinite degree". This is the property that gives power series their place on the Mount Rushmore of Calculus.

The essential thing from the above examples is that *everything* you could do to functions in order to build new, more complicated functions from simpler ones can be carried out with power series in the most natural way, as if they were polynomials. Therefore, once you have power series representing all the basic functions (and you do have them for most of the basic functions already), you can calculate from them power series for all those more complicated functions you construct from the basic ones by all kinds of combinations.

[1] The series should have the sum $\tan x$, and $\tan x$ goes to $\pm$infinity as $x \to \pm\pi/2$. On the other hand we know that in the interval of convergence of any power series, in particular this one, the sum will be a differentiable function. Therefore the points $x = \pm\pi/2$ cannot lie in the interval of convergence.

PS 4: Why "Radius" of Convergence, or, Why Non-Real Numbers Are the Real Thing; Honors Track

From the very beginning of our calculus course, we have never considered complex numbers. There does however exist a calculus for functions that accept complex numbers as arguments. In many respects, this version of calculus is quite different from the calculus you have learnt, one of the most modest differences being that you could not draw graphs any more and could not view the derivative as a slope of the tangent. So there is good reason not to drown you in complex variables in the first year. However, there are some features of calculus where complex numbers blend in very well, and it is more an (albeit very common) policy decision rather than a necessity to leave complex numbers out of these parts of calculus.

In particular, the arithmetic operations $+$, $-$, $\times$ work just as well with complex numbers. Moreover, limits of sequences of complex numbers can be handled just as well as limits of sequences of real numbers, with essentially no change. Therefore all ingredients are available to plug in complex numbers x into a power series and to ask the question whether the series converges for such x. The answer would continue to be the same: there is some R, called radius of convergence, such that the series converges for those x satisfying $|x-a| < R$ and diverges for those x satisfying $|x-a| > R$. Now if you remember that a complex number $x = u + iv$ can be represented as the point (u, v) in a plane, and that the absolute value $|x-a|$ denotes the usual distance between the points x and a in the plane, you see that the set of complex numbers x satisfying $|x-a| < R$ is actually a disc with center a, and R is its radius. This is why R is called *radius* of convergence.

And for that matter, if you choose to plug in complex numbers into power series, then you can see from (4) and (7) that $(e^{ix} - e^{-ix})/(2i) = \sin x$. *If you have ventured so far into strange territory already, you can just as well offer yourself a treat and try this calculation. It's not so difficult.* Note the similarity with $(e^{x} - e^{-x})/2 = \sinh x$. This similarity is the deeper reason why trig and hyp functions behave so similarly. Moreover, this curious formula just mentioned will be particularly relevant for electrical engineers at a more advanced level of their career. But this would be another long story to tell, and we really cannot give away the entire plot here already; we want your business for a few more semesters `;-)`

PS 5: Taylor Series

PS 5.1: Theory

We have studied power series and functions represented by them. Now we focus on the functions and on the question how to find a power series representing any given function. Actually, we started with this focus in section PS 1. Given the cosine function we constructed a power series that converges to $\cos x$. But this construction was cooked up specifically for sine and cosine, and we couldn't do the same thing with other functions. So here is the

Problem: Given a function f and a number a, can we find a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ centered at a that converges to $f(x)$ at least in some interval $(a-R, a+R)$?

The example (6) already alerts us that we may have to be content with convergence in an interval that is smaller than the domain of f; namely, in this example, $f(x) = \arctan x$ makes sense for every x, but the series converges only for $|x| \leq 1$. Moreover, we learn from section PS 3.1 that, if we can solve the problem, then f would automatically be differentiable in the interval of convergence. So clearly, we could not solve the problem with $f(x) = |x|$ and $a = 0$, because $|x|$ is not differentiable at 0. PS 3.1 can be used repeatedly, so we have to assume that not only f' exists, but also f'', f''' etc. Otherwise a power series as required in the problem couldn't exist. So we have the

Refined Problem: Given an *arbitrarily often differentiable* function f and a number a, can we find a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a that converges to $f(x)$ at least in some interval $(a-R, a+R)$?

The answer will be: "Typically yes, but in exceptional cases, it may be no." And this vague answer will be made more precise. By "typically", I am referring to those functions you will actually encounter in most practical situations. The exceptional cases are such functions that have been constructed for the sole purpose of getting a negative answer. Our approach to the problem has two steps: among all conceivable power series, we first weed out the slate of candidates until only a single power series remains, as the only candidate that has a chance to solve the problem. This candidate will then "be invited for an interview" to see whether it really solves the problem. Typically this "interview" will be successful and we get a solution, in the exceptional cases however, there will be no solution at all.

So let us look what a candidate $\sum c_n(x-a)^n$ must be like so that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots \tag{$*$}$$

is actually true for all x in a neighborhood of a. Well, in particular, the equation has to be true for $x = a$, and this means $f(a) = c_0$. So we want c_0 to be $f(a)$; any other choice would already make the candidate fail the first test. Now if (*) holds for all x in some neighborhood of a, we can take the derivative of this equation and obtain

$$f'(x) = \sum_{n=1}^{\infty} c_n n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots \tag{$*'$}$$

In particular, $f'(a) = c_1$. So we know also what c_1 has to be for the successful candidate. Next, from $f''(x)$ you calculate c_2, and in general, you will find that $c_n = f^{(n)}(a)/n!$. So we have narrowed down the field of candidates to only a single one: The only power series that has a chance to solve the problem is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We will call this only remaining candidate the Taylor series for $f(x)$ at a. Next, we have to answer a twofold question:

Questions: Given an arbitrarily often differentiable function f and a number a, (1) does the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ converge and (2) if so, does it converge to $f(x)$?

In typical cases, the answer to both questions will be yes, but for some exceptional examples, either question may have a negative answer. To decide the right answer in an individual case, we need to check whether (for which x if at all) $f(x) - P_N(x) \to 0$ as $N \to \infty$, where P_N is the N^{th} partial sum of the Taylor series. Without proof, you may use the following formula:

$$f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{(x-a)^{N+1}}{(N+1)!} f^{(N+1)}(c) \qquad \text{for some } c \text{ between } a \text{ and } x. \tag{10}$$

You couldn't know the value of c (and c can even depend on N), but you won't need any more knowledge about c except that it is between a and x. You should select a worst-case scenario, i.e., find some expression M depending on x, a and N, but not on c, such that M exceeds $|f^{(N+1)}(c)|$ for any possible c and $\frac{(x-a)^{N+1}}{(N+1)!}M$ still goes to 0 as $N \to \infty$. If you are successful in this, you will have shown that the Taylor series actually converges to $f(x)$.

Instead of undergoing the trouble to understand a proof of (10), try a much simpler homework: Show that for $N = 0$, equation (10) reduces to the mean value theorem. (And look up the mean value theorem, if you don't remember it.) (10) can actually be viewed as an elaborately brewed version of the mean value theorem.

PS 5.2: Series for the Exponential and the Square Root, A Do-It-Yourself-Cookbook

Close your notes and your textbook, then find the Taylor series for e^x centered at $x = 0$. Use formula (10) to show that this series converges indeed to e^x.

We have already seen a completely different method (specific to this particular case) to show that this Taylor series converges to e^x, a few pages above under 'Application'.

Now let's finally finish (9). Actually, this is also a very nice practice problem for you.

Take $f(x) = \sqrt{1-x}$, write down, in an orderly way, $f'(x)$, $f''(x)$, $f'''(x)$ etc until you recognize the general pattern. This allows you to write a formula for $f^{(n)}(x)$. Now write down the Taylor series of $\sqrt{1-x}$ at $x = 0$ (the answer will be (9), so you can check your result). Now you use formula (10) to show that the series actually converges to $\sqrt{1-x}$ for $|x| < 1$. Don't bother to prove convergence for $x = \pm 1$ by that same method as well. Instead refer to the appropriate paragraph in these notes to explain why the series still converges for $x = \pm 1$, even though you didn't bother to do a calculation for that case.

This will have handled (9). And to crown your success, you can now also find the Taylor series for $\arcsin x$ as follows: *Use (9) to find the Taylor series at 0 for $1/\sqrt{1-x}$. No, don't do a long division of power series. Use a different technique from PS 3.x to get from $\sqrt{1-x}$ to $1/\sqrt{1-x}$. Then write down a power series for $1/\sqrt{1-x^2}$. And now I won't tell you the final step, because I hope you will see it, if necessary after discussion with a colleague*

PS 5.3: The Mouth Dog and the Tail Dog

It's a common question: What is the difference between a power series and a Taylor series? The answer is simply: none. To be more precise: If you call a dog that is heading towards you a mouth dog, but a dog that is walking away from you a tail dog, then the difference between power series and Taylor series is the same as the difference between a mouth dog and a tail dog. The dog is the same, the difference is in the point of view.

First point of view: start with a mouth dog, i.e., a convergent power series $\sum c_n(x-a)^n$. This power series would just be called power series, because it is a power series. There is no function f around yet, so we wouldn't call the series a Taylor series, because a Taylor series $\sum \frac{f^{(n)}(a)}{n!}(x-a)^n$ will always belong to some function f, namely the one whose derivatives at a enters into the Taylor series. (I'll call this function the parent function of the Taylor series, but this is not an official term.)

With only the power series $\sum c_n(x-a)^n$ around, we do not have a function f yet. But if our series converges (on some interval $(a-R, a+R)$), we get one immediately: convergence means we may actually plug in numbers for x into the series and get a number as a result. This defines a function: $f(x) := \sum c_n(x-a)^n$. (f is arbitrarily often differentiable.)

Now we are at the second point of view, we have some (arbitrarily often differentiable) function f. Then we can write down its Taylor series $\sum \frac{f^{(n)}(a)}{n!}(x-a)^n$ (a tail dog) and study whether it converges, and if so, what it converges to. Let's do this for the function f from the previous paragraph. The definition is good for quickly calculating $f(a) = c_0$, $f'(a) = c_1$, $\ldots$, $f^{(n)}(a) = n!c_n$. Using this, we see that our Taylor series (aka tail dog) is actually $\sum \frac{n!c_n}{n!}(x-a)^n = \sum c_n(x-a)^n$. So the tail dog is the same fellow as the mouth dog, but we have approached it from the opposite direction. And this is why we won't need much work with formula (10) to check that the Taylor series converges to $f(x)$. We know it already from the very beginning, and we have just completed a small round trip. (Blessed are those who go around in circles, for they shall be known as big wheels, but you know this already from a previous manuscript.)