37. The parabolas $y = (x-2)^2$ and $x = (y-2)^2$ intersect to form a 'quadrangle' with curved sides. Find the coordinates of the intersection points. (Two can be guessed, the other two need calculation). Then calculate the area of this 'quadrangle'. Make sure to split the integral appropriately, depending on which formulas apply for the 'upper' and 'lower' boundary curves.

38. A cognac glass has the shape of a "sphere with a cap cut off". See figure for a cross section. If the radius of the ball is R and the glass takes up three fourths of the vertical diameter, what is the volume? (Yes I know these glasses are not meant to be filled to the brim, but calculate the total volume anyways.)

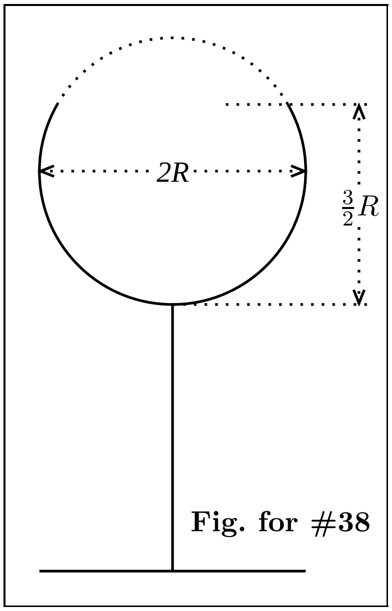


Fig. for #38

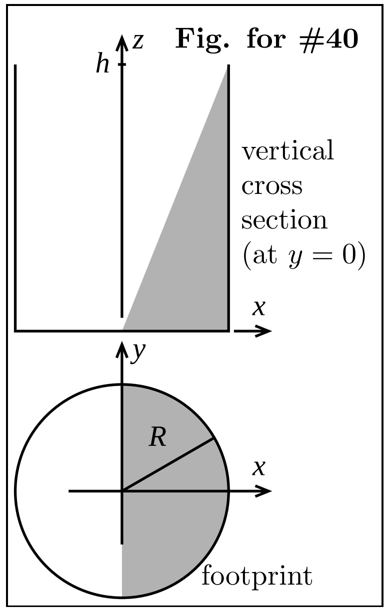


Fig. for #40

39. Two cylinders, each with a circular cross section of R intersect in such a way that their axes meet at a right angle. What is the volume of the intersection body? I find it difficult to draw, but Rogawski (Sec 6.2, Hwk 21 of the 2008 edition) has a figure of it, and also some hints.

40. A glass has the shape of a circular cylinder. It is initially full with water. We pour out some water carefully by leaning the glass sideways until the surface of the remaining water touches the bottom of the glass along a diameter. What percentage of the water remains in the glass?

Assume the radius is R and the height h, but express the remaining volume as a fraction of the total volume $\pi R^2 h$.

Note/Hint: There are at least three ways of doing the problem. One in which the slices are rectangles, one in which the slices are right triangles, and one in which the slices are segments of a circle. Set up integrals for all three of them.

Evaluate the integral obtained in at least two of the ways.

The figure above shows various sections of the glass, which is standing, with the water (grey) 'frozen' in place, centered at the origin of the x–y–plane. The vertical axes is the z axis.

Rectangles: slices parallel to the y–z–plane, different slices have different x.
Triangles: slices parallel to the x–z–plane, different slices have different y
Circular segments: slices parallel to the x–z–plane, different slices have different z

41. (a) Given a torus that is obtained by rotating a circle of radius r, centered at $x = R$ on the x–axis, about the y–axis, use the shell method to calculate its volume. (The answer should of course coincide with the answer $V = 2\pi^2 r^2 R$ obtained by slicing in class.)

(b) Now assume $R < r$, so the circle that rotates about the y axis intersects the y–axis. We only cosider the part of the circle that is in $x \geq 0$, let it rotate about the y–axis. Use the shell method to find the volume of the rotation body. [Note that in this case, the slicing method from class would be more cumbersome, since we would have washers for some y and disks for other y.]

42. The area below the curve $z = e^{-x^2}$ and above $z = 0$ rotates about the z–axis. Calculate its volume by means of shells. Note: You can expect to get a finite volume, even though the body extends all the way to ∞. The volume is to be understood here as an improper integral, i.e., you first consider only the area from $x = 0$ to $x = R$ rotating, and the volume you obtain this way. Then you let $R \to \infty$.

43. Let's go back to volume by slicing and do the previous problem that way. We have the body above the x–y–plane and below the rotated version of the curve $z = e^{-x^2}$, which is the surface $z = e^{-r^2}$, where $r^2 = x^2 + y^2$. Slice the body by planes perpendicular to the y–axis. You will run into the difficulty that you cannot evaluate $\int_{-\infty}^{\infty} e^{-x^2}\, dx$. Just call this unknown number I. Its numerical value is $I \approx 1.8$.

Now express the volume of the said body as a simple expression involving the quantity I.

Compare your result with the previous problem. From this comparison, conclude the *exact* value of $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$.

[So here you have an example where you find the value of a definite integral without going through the process of finding an antiderivative, where an antiderivative is actually not available to you in this example. — There are some advanced methods to find definite integrals without antiderivatives, like such as are based on complex-variable techniques (Math 443); but apart from those advanced methods, available examples are unique and based on individual ad-hoc tricks.]

Knowledge of the value of this integral is desired, due to its frequent occurrence in probability ('bell curve') and sometimes physics.

44. Calculate the perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as an integral $2\int_{-a}^{a} \ldots\, dx$. (You will not be able to evaluate this integral, so leave it alone, except for algebraic simplification of the expression under the integral.)

Use the trig substitution $x = a \sin u$ to convert the obtained integral into another integral (which you still won't be able to finish up, but which may turn out to be more convenient in some cases, like for numerical purposes).

[Note: Integrals containing trigs under a square root (in not too complicated a manner), or 3rd or 4th degree polynomials under a square root, or the ratio of two quadratics under a square root, are known under the label 'elliptic integrals' for the very reason that they show up when calculating the perimeter of an ellipse. It is good to recognize them and to know that they will *not* be evaluated by Calc 2 methods at all, unless you can get rid of either the trig or the square root by simple algebra or a simple u-substitution. These integrals do show up in applications. If you recognize them, you save time by *not* trying your antiderivative prowess on them in vain; they are well-studied, and mathematics has a plethora of wisdom about them if you are desperate enough to ask for it.]

45. (a) Calculate the length of the arc of the parabola $y = ax^2 - 1$ between $x = 0$ and $x = b$. [The '-1' part is irrelevant here and just offered for convenience in part (b).] You may borrow a good deal of work from either 27d or 29d, after doing either a trig or hyp substitution to finish up the integral.

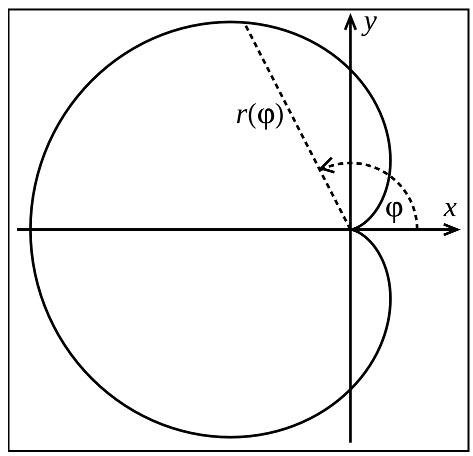
The 'limit' part of b is voluntary; but if you skip it you have to do part c instead. Part c is much easier, but way less fun.

(b) A downsized version of a Putnam competition problem in 2001 asks whether it is possible to choose a in such a way that the part of the parabola inside the disc $x^2 + y^2 \leq 1$ is longer than 4. Assume $a > \frac{1}{2}$ and find the appropriate intersection point $(b, ab^2 - 1)$. Calling $L(a)$ the length of that part of the parabola inside the disk, as a function of a, calculated according to part (a), show that $a(L(a) - 4)$ goes to ∞ as $a \to \infty$. Conclude the correct answer to the Putnam question.

(c) Instead of calculating the limit as $a \to \infty$ in part (b), use a pocket calculator to evaluate the length of the piece of the parabola inside the disk in the case $a = 100$. Answer the Putnam question. FYI: there are no calculators on the Putnam competition.

46. Take the ellipse from problem 44, let it rotate about the x-axis. Find the surface area of the body thus obtained. You may assume $b < a$. Note: This time, evaluation of the integral is quite manageable.

47. For the torus from #41a, calculate its surface area.



48. The function $r(\varphi) = a(1 - \cos\varphi)$ describes a heart–shaped (or apple–shaped?) curve called the cardioid.
(a) Calculate the area A it encloses;
(b) also calculate the length L of this curve.
(c) As a simple consistency check, confirm the isoperimetric inequality, namely that $A/L^2 < \pi/(2\pi)^2$.

49. (a) That same cardioid now rotates about the x–axis. Calculate the surface area S of the apple–shaped body thus obtained.

(b) Also calculate the enclosed volume V. **Hint:** Slices in the form of disks or washers may actually be easiest. Note that $x = r(\varphi)\cos\varphi$ and $y = r(\varphi)\sin\varphi$ and use the formula $\int_{x_{min}}^{x_{max}} \pi y^2\,dx + \int_{x_{max}}^{0} \pi y^2\,dx$ (explain why?), with x, y, dx properly expressed in terms of $r(\varphi)$ and φ.

(c) Confirm that your results are consistent with the isoperimetric inequality $V^2/S^3 < (\frac{4}{3}\pi)^2/(4\pi)^3$.

50. You sip a drink from a cocktail glass with a straw. The cocktail glass is rotationally symmetric with a vertical cross section given by $y = x^2$ (where $|x| \le 3$). The straw extends 6 unit lengths (meaning 6cm) above the rim of the glass. Assuming the density to be 1.01937 (meaning 1.01937 g/cm^3, as even juice is mostly water, plus a bit of sugar), and no ice (horror, that must have happened on a study abroad, but it does simplify the math;-), what is the work it takes yu to empty the glass by this method, using that 1 Joule = 1 Nm and N=m kg/sec^2? Use the acceleration of gravity to be 9.81 m/sec^2.

I made up the density in such a way that $1.01937 \times 9.81 = 10$ for numerical convenience

51. Here you will numerically evaluate the integral $\int_0^1 \sqrt{(1-x^2)\Big(1-\frac{x^2}{2}\Big)}\,dx$. (Recall the comment made in assignment #44 that this is an integral not amenable to the algebraic techniques of Calc 2 — albeit well-studied in advanced mathematics).

(a) Use each of trapezoidal rule, midpoint rule, and Simpson's rule while splitting the interval of integration into four subintervals of equal length, comparing the results with 6 digits behind the decimal point.

(b) Calculate the 2nd derivative of the integrand and show that it is negative for all x between 0 and 1. *Hint: Combining all terms, you may get a numerator* $-6+12x^2-9x^4+2x^6$. *Be a bit inventive with the algebra to determine its sign when* $0 < x < 1$. *If you don't see a good idea use that* x to power six is always less than one, to get rid of that one term. Explain why the error estimates (relying on M_2 or M_4 such that $|f''(x)| \le M_2$ or $|f''''(x)| \le M_4$) for each of the three rules given in the book are useless in this case. Also give explicit upper and lower bounds for the integral based on your knowledge that $f''(x) < 0$.

(c) Now for comparison, substitute $x = \sin u$ and use each of trapezoidal, midpoint and Simpson rule on the new integral, again with four subintervals of equal length, again giving 6 digits behind the decimal point in each case. Give error margins for the results, based on the formulas; finding bounds M_2 and M_4 for the second and fourth derivatives of the integrand is a bit clumsy: you may instead simply use the best values found by means of symbolic algebra and graphing software, namely $M_2 = 2.5$ and $M_4 = 15.25$.

(d) Observe the improvement in precision when comparing (a) and (c). Compare the result for trapezoidal and Simpson rules with 10 equidistant intervals (and the respective error margins obtained). *[Just to save labor, I am skipping midpoint in this part.]*

52. *"How large is the number '1000!' ?" Can we get a simple formula that allows us to give a good approximation for n! for large n that does not require to go through all the multiplications one by one?*

Integrals can help in this task. We can compare $\ln n! = \ln 1 + \ln 2 + \ln 3 + \ln 4 + \ldots + \ln n$ with an integral that we can actually calculate. So, just for a change, in this problem we will use (easy) integrals to give a good approximation for (messy) finite sums, rather than using easy finite sums to find an approximation for messy integrals!

Task 1: Use the midpoint rule with $\Delta x = 1$ on the integral $\int_{3/2}^{n+1/2} \ln x\,dx$ to show that

$$\ln n! \ge (n+\tfrac{1}{2})\ln(n+\tfrac{1}{2}) - (n+\tfrac{1}{2}) - \text{some number}$$

(of course you want to calculate this "some number" precisely!) Sketch a graph and observe that the correct sign of the 2nd derivative of the logarithm function tells you why you have '$\ge$' and not '$\le$'.

Task 2: Use the trapezoidal rule (with $\Delta x = 1$) on the interval from 2 to n to show that

$$\ln n! \le \tfrac{1}{2}\ln(2n) + n\ln n - n + \text{some number}$$

(again you want to calculate this number exactly...) Sketch a graph and observe, from the sign of the second derivative of the logarithm function, why you get '$\leq$', not '$\geq$'.

Task 3: The $\ln(n+\frac{1}{2})$ in the result of task 1 is just a bit un-beautiful, we'll prefer $\ln n$: Explain with a 1-term Riemann-sum why $b\int_a^b \frac{dx}{x} > (b-a)$ when $b > a > 0$. Use this to show (by convenient choice of a, b) that $(n+\frac{1}{2})\ln(n+\frac{1}{2}) > (n+\frac{1}{2})(\ln n) + \frac{1}{2}$.

Combine this result with your result from Task 1 to show

$$\ln n! \geq (n+\frac{1}{2})\ln n - n + \text{some number}$$

where again you want that number precisely.

Task 4: Combine your results and apply the exponential function to them to find that

$$C_1\sqrt{n}\,\frac{n^n}{e^n} \leq n! \leq C_2\sqrt{n}\,\frac{n^n}{e^n}$$

where we want the C_1 and the C_2 both as exact numbers and numerically with 3 decimals.

Task 5 for you is only to watch in amazement, you don't have work of your own here: Advanced Calculus methods that go beyond what we can do here improve on the result from Task 4; they actually show that

$$\frac{n!}{\sqrt{n}\,n^n/e^n} \to \sqrt{2\pi} \quad \text{as} \quad n \to \infty$$

The numerical value of $\sqrt{2\pi}$ with 3 decimals is 2.507, which should be neatly between the C_1 and C_2 obtained above.

Yet another occurrence of π in a context that has nothing to do with circles!

53. (a) Determine, possibly depending on x, whether the series $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ converges or diverges.

(b) For which x does the series $\displaystyle\sum_{n=0}^{\infty}\frac{x^{n^2}}{n!}$ converge? (While we are discussing only series with positive terms, you may pretend $x > 0$.)

(c) Determine, depending on x, whether the series $\displaystyle\sum_{n=0}^{\infty}\frac{n!^2x^n}{(2n)!}$ converges. (Again, you may presume $x > 0$ for now. And you may leave a borderline case, where ratio or root test is inconclusive, undecided.)

54. Apply the ratio and the root tests to the p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$. Do you get a useful answer?

55. Recall from a prior problem that there are certain positive constants C_1 and C_2 (namely $C_1 = 2.43$ and $C_2 = 2.62$ would work) such that $C_1\sqrt{n}\,(\frac{n}{e})^n < n! < C_2\sqrt{n}\,(\frac{n}{e})^n$. This will help you, as the question here is to decide the convergence of $\sum \frac{x^n}{n!}$ by means of the *root* rather than the ratio test. (For the time being, as we study series with positive terms only, you may pretend $x > 0$.)

56. Does $\displaystyle\sum_{n=1}^{\infty}\frac{n^{1000}}{3^n}$ converge? Use ratio test, root test and limit comparison (choosing an appropriate partner series to compare with).

57. (a) Consider the series $1 + \frac{2}{3} + \frac{1}{3^2} + \frac{2}{3^3} + \frac{1}{3^4} + \frac{2}{3^5} + \ldots$. The denominators are powers of 3, but the numerators alternate between 1 and 2. In $\sum$ notation, you could write this as

$$\sum_{n=0}^{\infty} \frac{(3-(-1)^n)/2}{3^n} \qquad \text{or} \qquad \sum_{n=0}^{\infty} \frac{(3-\cos(\pi n))/2}{3^n} .$$

Show that the ratio test is inconclusive, but use either the root test or direct comparison to decide on whether the series converges
(b) Nothing to do here except to read & acknowledge: Is the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \ldots$ convergent or divergent? (The terms are the reciprocals of the prime numbers.) None of the methods seen so far or later in Calc 2 are good enough to decide this question. This goes to showing that one can easily invent series whose convergence is difficult to decide. But for the series that are useful in calculus, the decision can usually be made routinely.

(In case you really want to know: The sum of the reciprocals of prime numbers is known to be a divergent series.)

58. Consider $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$. Does this series converge or diverge? (Hint: Comparison with an integral is the way to go.) — Next decide $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ where $p > 0$ is any given real number. (The answer will depend on p.) Note: You learn more by simply following the method I used in class when deciding the p-series by comparison with an integral. Explain the situation with a figure using terms like 'left' or 'right endpoint' Riemann sum and 'decreasing'. — Once you have done this, reading up precisely on the highlight box about 'integral comparison' in the textbook will make more sense than it would if you were to just plug quantities into the highlight box paradigm.

59. (In continuation of #53c:) Use the result from Hwk #52 (requoted in #55), together with an appropriate comparison, to decide whether the series $\sum_{n=0}^{\infty} \frac{4^n\, n!^2}{(2n)!}$ converges or diverges.

60. We have seen how the ratio test can be used to show that $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ converges absolutely for every x (cf. #53a), and we have seen that the value of this series cannot be anything else but $\cos x$ because partial sums are alternatingly above and below $\cos x$ (cf. #26).

(a) Explain why Leibniz' alternating series test applies to this series for $x = 1$, but does not apply to it for $x = 2$.
(b) For which x exactly does the alternating series test apply to the cosine series?

61. Based on the knowledge that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, write a series for e^{-x} and then conclude that the series in #53a converges to $\cosh x$. *Hint: Write out the sums with the '…' notation rather than manipulating expressions involving $\sum_{n=0}^{\infty}$, as you may lack the experience to do the manipulation of $\sum_{n=0}^{\infty}$ expressions correctly/expediently.*

62. (a) What is the radius of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n\, n!^2}$?
(b) The series defines a function $J(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n\, n!^2}$. Since you won't be able to find a formula for the value of the series, you know practically nothing about this function. Calculate $J(1)$ with to 3 decimals accuracy. How many terms did you need? How many terms would you need to calculate $J(2)$ with 3 decimals accuracy? Calculate $J(0)$, $J'(0)$, $J''(0)$, $J'''(0)$ and $J^{(4)}(0)$ exactly.

63. Using the geometric series $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=0}^{\infty} t^n$ and the substitution $t = -x^2$ to find a power series representation for $\frac{1}{1+x^2}$. Integrate it to obtain a power series representation of $\arctan x$. What is the radius of convergence of the obtained series? (In each step, write the power series both in a '. . . ' form giving enough terms to see the general pattern and using the $\sum$ notation.)

Substitute $x = 1$ into the arctan series (believing me that doing so is permissible). Obtain a beautiful but useless series representing π. How many terms would you need to calculate in order to obtain π up to an accuracy of 10^{-4} by this series?

Calculate $\arctan \frac{1}{5}$ to 5 decimals using this series. (Of course in doing so you pretend your pocket calculator does not know arctan.) How many terms do you need to do this?

Also calculate $\arctan \frac{1}{239}$ to 5 decimals. How many terms do you have to use for this purpose? [We'll later see why, of all crazy numbers, I would specifically care for 1/239.]

64. Write down the power series representing $\ln(1 + x)$ and $-\ln(1 - x)$ respectively (obtained by integrating a geometric series.) What is the radius of convergence?

Obtain a power series for $\ln \frac{1+x}{1-x}$ from the preceding two. Which is the radius of convergence for each of these series?

Use the series for $\ln(1 + x)$ and the one for $\ln \frac{1+x}{1-x}$, each with a specific choice of x, to obtain a series with value $\ln 2$. [You need to take my word for it that the choice of x in the first series is actually permissible.]

One of the two series is almost useless for practical calculation of $\ln 2$, because it would take way too many terms for decent precision. In the other one, roughly how many terms do you estimate you'd have to calculate to get 5 digits precision for $\ln 2$?

65. By multiplying the corresponding power series, obtain a power series representing $e^x\, sinx$. In doing so calculate all powers up to x^7; you are not expected to see a general formula for the nth coefficient, and so you will not be able to use the $\sum$ notation; just calculate the first few terms in the series as indicated.

66. Write down power series for $1 + \ln(1 + 2x)$ and $1 + \ln(1 + x)$ up to the x^4 term. By long division, obtain a power series for $\dfrac{1 + \ln(1 + 2x)}{1 + \ln(1 + x)}$, also up to the 4th term.

67. It is possible to obtain the well-known formula $e^x\, e^y = e^{x+y}$ directly by multiplying the power series, without using any prior knowledge about the exponential function. By multiplying the power series $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$ with the series $1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\frac{y^4}{4!}+\ldots$ and applying the binomial formula, verify that the product is indeed $1+(x+y)+\frac{(x+y)^2}{2!}+\frac{(x+y)^3}{3!}+\frac{(x+y)^4}{4!}+\ldots$, at least up to power 4.

Note: If you know the general binomial theorem in the form $(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}$, you can obtain the analogous calculation for *all* terms, not just up to the 4th power. However, for this homework, we'll be content with checking the result only up to power 4.

68. Recalling the power series of $\sin x$, and of $\cos y$, plug the series of $\sin x$ for y into the series for $\cos y$, calculating terms up to order x^6. This way you will obtain the beginning of the power series of $\cos(\sin x)$.

69. Calculating the first five derivatives of $\tan x$ and using Taylor's formula, find the first 3 nonvanishing terms (i.e., up to order x^5) for the Taylor series of $\tan x$, and compare it with the series obtained in class by long division. *Hint: It's probably more convenient for successive derivatives if you write the derivative of* $\tan x$ *as* $1+\tan^2 x$ *rather than* $1/\cos^2 x$.

70. The nice features of term-by-term differentiation being allowed for power series do not carry over to other series. For instance $f(x) := \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is a useful series that is *not* a power series.

(a) Show by a direct comparison test, that this series converges absolutely for every real x.

(b) Attempt to take a second derivative via term-by-term differentiation (twice). What series do you get as a result? Does it converge for any x ? (A somewhat heuristic answer is acceptable here, even if it is not logically watertight.)

Note: You have no tools at this level to guess the value of this sum $f(x)$. Senior level mathematics would tell you that actually $f(x) = \frac{\pi^2}{6} - \frac{\pi}{2}|x| + \frac{1}{4}x^2$ for $|x| \leq \pi$. — The only reason why you see this example in Calc 2 is to convince you that some of the 'nice' features of calculation with power series are not 'automatic' carry-overs from the rules for finite sums.