Homework
UTK – M148 – Honors Calculus II – Spring 2015
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1. Find antiderivatives of the following functions:

$$(a) \quad f(x) = \cos(2x+4) \qquad\qquad (b) \quad g(x) = 3e^x + e^{3x} + x^{3e}$$

$$(c) \quad h(x) = \frac{1}{5x} \text{ for } x > 0 \qquad\qquad (d) \quad k(x) = 1/x^5$$

2. Find the area below the curve $y = x^2$, above the x-axis, between $x = 0$ and $x = b$ as a limit of a sum of rectangle areas (as done for $y = ax$ in class). To this end, divide the interval $[0, b]$ into n pieces each of equal length $\Delta x = \frac{b}{n}$. Do two calculations, one by evaluating the function at the left endpoint, one at the right endpoint.

Draw a picture.

Hints: You may need the following formula (which is likely new to you, and you may just take my word for its validity): $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. You may also need a similar formula ending with $(n-1)^2$, which you can obtain from the one I gave you by simple algebra.

3. (a) Evaluate $\sum_{j=0}^{6} j(6-j)$.
(b) Simplify the expression $\sum_{j=1}^{n} f(j) - \sum_{k=2}^{n-1} f(k)$ (where you may assume that n is an integer ≥ 3).
(c) Evaluate $\sum_{j=1}^{n} \frac{1}{j(j+1)}$ for $n = 1, 2, \ldots, 7$ respectively and conjecture a formula for general n. Then write $\frac{1}{j(j+1)}$ as a difference of two terms, based on the conjectured formula, and use this to prove the formula as a telescoping sum.

4. Estimate $\int_0^5 \frac{dx}{1+x^2}$ from above and below by a Riemann sum each; make the Riemann sums consist of 10 terms (using an equidistant partition).

5. We'll get a 4-digit precise result for $\int_0^3 \frac{dx}{x^2+3}$ here. You will like a programmable calculator for this, but the main work is still analytic work on the paper. Here is the philosophy of this problem: While Riemann sums are good for theoretical purposes, a practical calculation will give more precise results by using trapezoids rather than rectangles.

(a) Calling $f(x) := 1/(x^2+3)$, find out on which subinterval of $[0, 3]$ we have $f''(x) > 0$ and $f''(x) < 0$ respectively. Answer should be in the form "$f''(x) < 0$" if $x <??$ and $f''(x) > 0$ if $x >??$".

(b) Instead of nesting a small slice of area beneath the graph of f, from x_j to x_{j+1}, between rectangles, nest it between trapezoids: One trapezoid will have the oblique line connecting $(x_j, f(x_j))$ and $(x_{j+1}, f(x_{j+1}))$, the other trapezoid will have the oblique line being a tangent to the graph of f at $x = \frac{1}{2}(x_j + x_{j+1})$. Which trapezoid gives a lower bound for the area, which gives an upper bound, and how does the answer depend on issues discussed previously?

(c) Write out a formula for the areas of the individual trapezoids involved in the calculation, then sum up the appropriate areas to get an upper and a lower bound for $\int_0^3 \frac{dx}{x^2+3}$: choose 150 subintervals of equal length in $[0, 3]$. Make sure that you select the midpoint rule for some intervals and the trapezoidal for others, as discussed in (a) to get either an upper or a lower bound for the integral. Now you may want to use technology and get actual numerical values.

6. Evaluate the following integrals (among the expected answers, there may be: "cannot do it with tools available", or, "integral may not exist since integrand isn't piecewise continuous"):

$$(a)\quad \int \cos x\,dx \qquad\qquad (b)\quad \int \big(3x^3 - 4\sin(2x)\big)\,dx$$

$$(c)\quad \int_2^7 \frac{1}{x^2}\,dx \qquad\qquad (d)\quad \int_1^5 \frac{\sin x}{x}\,dx$$

$$(e)\quad \int_{-2}^2 \frac{1}{x^2}\,dx \qquad\qquad (f)\quad \lim_{x\to\infty}\int_1^x t^{-3/2}\,dt$$

$$(g)\quad \int_0^1 x^2\,dx \qquad\qquad (h)\quad \int_0^1 t^2\,dx$$

7. Let

$$f(x) := \begin{cases} x^2 & \text{if } |x| < 1 \\ 2|x| - 1 & \text{if } |x| \geq 1 \end{cases}$$

Find $\int_{-1}^3 f(x)\,dx$

8. Take the same function f as before. Calculate $F(x) := \int_0^x f(t)\,dt$. The answer should be a formula that also involves if's. Graph both f and F on the interval $[-2, 2]$ in the same coordinate system. (In this graphing job, a halfway decent handmade figure, like what I'd draw on the blackboard, is preferable to a print-quality technology-generated figure.)

9. Find the following derivatives without attempting to evaluate the integrals first.

$$(a)\quad \frac{d}{dx}\int_0^x \sin^2 t\,dt \qquad\qquad (b)\quad \frac{d}{dx}\int_x^{50} (\sin^2 t)/t\,dt$$

$$(c)\quad \frac{d}{dx}\int_x^{2x} \frac{dt}{1+t^4} \qquad\qquad (d)\quad \frac{d}{dx}\int_x^{2x} \frac{dt}{t}$$

$$(e)\quad \frac{d}{dx}\int_0^{\sin x} \frac{\sin t}{t^2+1}\,dt \qquad\qquad (f)\quad \frac{d}{dx}\int_0^{x^2} g((t+1)^2)\,dt$$

10. Of the following three integrals, two are prohibitively difficult, one is easy. Select the easy one and calculate it:

$$(a)\quad \int \sqrt{2+\sin x}\,dx\,, \qquad (b)\quad \int \cos x\sqrt{2+\sin x}\,dx\,, \qquad (c)\quad \int \sin x\sqrt{2+\sin x}\,dx$$

11. Calculate $\displaystyle\int_1^2 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

12. Calculate (a) $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx$, (b) $\displaystyle\int_2^4 \frac{1}{x\,\ln x}\,dx$.

13. Evaluate $\displaystyle\int_0^8 \sqrt{1+\sqrt{1+x}}\,dx$.

14. Find $\int \tan x\,dx$. If, after giving it some thought, you need a hint: tangent is sine divided by cosine

15. Find $\displaystyle\int \frac{2dx}{e^x + e^{-x}}\,dx$, which is the same as $\displaystyle\int \frac{dx}{\cosh x}$.

16. (a) Calculate $\int x^3e^{-x}\,dx$.
(b) Given any positive integer n, express $\int_0^N x^ne^{-x}\,dx$ in terms of $\int_0^N x^{n-1}e^{-x}\,dx$. Then express $\lim_{N\to\infty}\int_0^N x^ne^{-x}\,dx$ in terms of $\lim_{N\to\infty}\int_0^N x^{n-1}e^{-x}\,dx$.
(c) Evaluate $\lim_{N\to\infty}\int_0^N x^{100}e^{-x}\,dx$. Rather than writing the result of a huge integer with over 150 digits, write it as a product of many 1- and 2-digit numbers without further evaluation.

17. Evaluate $\int e^{2x}\cos 3x\,dx$.

18. Calculate $\int_a^1 x^2(\ln x)^2\,dx$ for $a>0$. Then calculate $\lim_{a\to 0+}\int_a^1 x^2(\ln x)^2\,dx$ from it.

19. Calculate $\int_0^4 e^{-\sqrt{x}}\,dx$.

20. Calculate $\int \arcsin x\,dx$

21. Calculate $\int \frac{\ln(\ln x)\ \ln x}{x}\,dx$

22. Calculate $\int_1^4 \sqrt{x}\,\ln x\,dx$

23. There once was a student named Mike Ivan Shap (as a Southerner, he went by his initials M.I.) who had forgotten the antiderivative of $\frac{1}{x^2+1}$. So he attempted integration by parts, using $u'=1$ and $v=\frac{1}{x^2+1}$.

Task 1: Carry out M.I. Shap's calculation and see what integral he obtained.

Of course he couldn't do the new integral, so he went to his teacher for help and received the reminder that integration by parts wasn't needed and in actuality, he should have remembered that...

Task 2: (too easy)

$$\int \frac{dx}{x^2+1} = ???$$

He felt frustrated and embarrassed about his forgetfulness, but fortunately his girl friend Miss Happy would always encourage him. She pointed out that he could use his unsuccessful attempt for finding a *new* integral; one that he would have been hard pressed to solve from scratch, had it been given without hint.

Namely she pointed out a simple piece of algebra:

$$\frac{x^2}{(x^2+1)^2} = \frac{x^2+1}{(x^2+1)^2} - \frac{1}{(x^2+1)^2}$$

Comment from your professor: I call a piece of algebra a $\int$implifiation, if it simplifies the purpose of doing an integral. Note that sometimes, a $\int$implification goes in the opposite direction of what you'd have been asked to do in an algebra course. The task in $\int$implification is to get the expression in a form that is convenient for doing some integral, whereas a simplification in a simple college algebra course wants you to get the expression in a form that (kind of) uses the least amount of ink to write down.

On another note, I remind you of the trick of going around in circles (and be known as a big wheel for it) as exemplified in the lecture and in the textbook.

Task 3: Evaluate $\displaystyle\int \frac{dx}{(x^2+1)^2}$ according to Miss Happy's advice.

Task 4: Now evaluate $\displaystyle\int \frac{dx}{(x^2+a^2)^2}$.

M.I. Shap and Miss Happy went to the movies to watch the 1968 movie 'The Thomas Crown Affair' (they are too classy to bother with the late 90's remake). Without spoiling much, a hinge point is Thomas Crown's quote: "I have done it once, I can do it again" with the implied challenge 'Watch me do it'. After this quote from the movie, they were inspired to tackle the following math problem together:

Task 5: Evaluate $\displaystyle\int \frac{dx}{(x^2+1)^3}$

24. Sometimes, definite integrals can be determined without finding an antiderivative: (a) Find $\int_{-7}^{7} \frac{\sin x\,dx}{x^4+1}$ without attempting to obtain an antiderivative. Give a quick sketch of the graph of the function under the integral sign to illustrate your reasoning. — (b) Use the substitution $y=-x$ on the integral to confirm your conclusion algebraically.

25. (a) Describe the graph $y=\sqrt{25-x^2}$ in geometric terms. Based on this deliberation, what should $\int_{-5}^{5}\sqrt{25-x^2}\,dx$ be? — (b) Use an appropriate trig substitution to verify your conclusion calculationally.

26. You sure know that $\cos x \le 1$ for all x, and probably also that $\sin x \le x$ for $x>0$. But you may not know yet that $\cos x \ge 1-\frac{1}{2}x^2$ for all x, and that $\sin x \ge x-\frac{1}{6}x^3$ for $x>0$. In this homework, you'll see why this is the case (and a bit more).

Recall: If $f(x)\le g(x)$ for $x\in[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

(a) Integrate the inequality $\cos t \le 1$ for all t over the interval $[0,x]$, with $x>0$. What inequality do you obtain?

(b) Next integrate the inequality so obtained (renaming the variable into t) over the interval $[0,x]$ again and solve for the trig function. What inequality do you now obtain?

(c) Repeat the procedure three more times, obtaining new inequalities for $\sin x$ or $\cos x$ in each step.

(d) Use the best of these inequalities to nest the values of $\cos\frac{1}{2}$ and $\sin\frac{1}{2}$ between values easily calculated by hand; i.e., find rational numbers for the '?' to make $?_1 \le \cos\frac{1}{2} \le ?_2$ and $?_3 \le \sin\frac{1}{2} \le ?_4$ true. None of your calc's will require a pocket calculator (except possibly for converting fractions to decimals in the very end).

27. Reduce the following integrals to integrals without a square root, by means of a trig substitution. In the problems with a *, you don't need to further deal with the thus obtained integrals (that will be a task we do later). In the other ones, finish up the integration

$$(a)\quad \int \frac{dx}{(4-x^2)^{3/2}}\,, \qquad (b)\quad \int (4-x^2)^{3/2}\,dx\,, \qquad (c^*)\quad \int \frac{x^2}{\sqrt{x^2+1}}\,dx$$

$$(d^*)\quad \int \sqrt{x^2+x+1}\,dx\,, \qquad (e^*)\quad \int \sqrt{(x-1)(x-3)}\,dx\,, \qquad (f^*)\quad \int_1^4 \frac{x}{1+\sqrt{x^4+1}}\,dx$$

$$(g)\quad \int \frac{\arcsin x}{x^2}\,dx\,, \qquad (h)\quad \int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$$

In (e) you may assume $x>3$. Hint for (f),(g): do something else (in each case: what?) before trying the trig substitution. Hint for (h): $\frac{1-x}{1+x}=\frac{(1-x)^2}{(1-x)(1+x)}$. Also to finish up (g) in the end, use that $\frac{1}{1-y^2}=\frac{1}{2}(\frac{1}{1-y}+\frac{1}{1+y})$.

28. In one of the examples of the previous problem, the trig substitution could be avoided, and a simple power substitution would work instead. Find which of them it is and also carry out the power substitution for comparison.

29. Redo all problems of the previous list (#27 a–h), using a hyperbolic substitution instead of the trig substitution. Subsequently get rid of the hyperbolics by expressing them as exponentials and substituting $e^t = y$ (assuming t is what you named the new variable in the hyp substitution.) If you end up with a rational expression in y that is *not* routine to integrate, you may stop here; we'll return to this task later. Otherwise, finish them up. If possible, draw a resume whether you feel the hyperbolic or the trigonometric substitution was easier.

30. (a) Reduce the integral $\int \frac{\sin^2 u}{\cos^3 u}\,du$ to the integral of a rational function in two ways: Either using the substitution $\sin u = y$ (taking advantage of the fact that one $\cos u$ can go with the differential, leaving only even powers of $\sin u$ and of $\cos u$). Or use $\tan \frac{u}{2} = t$, which is a universal tool working for all rational expressions of $\sin u$ and $\cos u$. Which of the two methods leads to the 'easier' rational integral (where 'easyness' is judged by the degree of the denominator)?

(b) Convert the integrals obtained in 27c-e into integrals of rational expressions.

(c) Next you will use the hint that $\frac{1}{1-y^2} = \frac{1}{2}(\frac{1}{1-y} + \frac{1}{1+y})$ and $\frac{y}{1-y^2} = \frac{1}{2}(\frac{1}{1-y} - \frac{1}{1+y})$, which is a piece of algebra that you will be able to invent yourself in a few weeks; and also a similar piece of algebra that you can obtain by squaring the given hints, like $\frac{1}{(1-y^2)^2} = \frac{1}{4}(\frac{1}{(1-y)^2} + \frac{1}{(1+y)^2} + \frac{2}{1-y^2})$ (then re-use the first hint for the last term!)

With these hints you should be able to finish up the integrals obtained in 27c,d,e. (Don't forget to undo the substitutions in the end to get answers in terms of x.)

31. Finish up #27f, using the universally helpful substitution for rational expressions in $\sin u$, $\cos u$ mentioned in the previous problem. (And I am not aware of a simpler procedure by means of a more specialized substitution). Make sure to get an exact expression first, in which all occurrences of trig(arctrig(number)) are simplified; then you may use technology to get a numeric result and check it for plausibility in view of the original integral.

32. Find the PFD of $\dfrac{5x^2+4x-13}{x^3+2x^2-5x-6}$ and calculate $\displaystyle\int_0^1 \frac{5x^2+4x-13}{x^3+2x^2-5x-6}\,dx$.

33. (a) Find the PFD of $\dfrac{(x+2)(x^2-2)}{(x+1)^2(x^2+1)}$ — (b) Find the PFD of $\dfrac{x^6+76}{(x+2)^3(x-3)^2}$

34. Find the PFD of $\dfrac{25}{(x-1)^2(x^2+4x+5)}$ in two ways: one using complex numbers to deal with the quadratic, one without use of complex numbers.

35. Use PFD to evaluate $I(a,b;x) := \displaystyle\int_0^x \frac{dt}{(t^2+a^2)(t^2+b^2)}$, assuming $a, b > 0$ and $b \neq a$. Finally, calculate the limit $\lim_{b\to a} I(a,b;x)$ (which may require l'Hopital). — The purpose of this problem is to obtain $I(a,a;x) = \int_0^x dt/(t^2+a^2)^2$ by a different method; we trust that $I(a,a;x) = \lim_{b\to a} I(a,b;x)$, i.e., that the integral depends continuously on b. — Compare your result with Hwk #23 (Task 4).

36. Use the trig substitution $x = a\tan u$ on the integral $\int dx/(x^2+a^2)^2$ and evaluate it this way.