Partial Fractions

Partial fraction decomposition is a method of algebraically transforming rational functions into a certain standard form that makes integration a matter of routine. For instance, we would transform

$$\frac{x^3+x-3}{x^4+x^3-x^2+x-2} = \frac{x^3+x-3}{(x-1)(x+2)(x^2+1)} = \frac{-1/6}{x-1} + \frac{13/15}{x+2} + \frac{\frac{3}{10}(x+3)}{x^2+1} \tag{1}$$

The main task and main work consists of understanding why we write it this way, and how we can find this form. Once you have written the integrand in this way, integration reduces to doing standard integrals you have seen already. Namely,

$$\int \left(\frac{-1/6}{x-1} + \frac{13/15}{x+2} + \frac{\frac{3}{10}(x+3)}{x^2+1} \right) dx = -\frac{1}{6}\ln|x-1| + \frac{13}{15}\ln|x+2| + \frac{3}{10}\left(3\arctan x + \frac{1}{2}\ln(1+x^2)\right) + C$$

We are using partial fraction decomposition here as a tool to find anti-derivatives. However, this is by no means the only purpose for the method. In an introductory course on Differential Equations, the same method is used as a tool to find inverse Laplace transforms. The power of this technique lies in the fact that it reduces rational functions to a sum of simple constituent terms that makes certain qualitative features visible. Namely, let us informally call a 'singularity' of a rational function a point where the function is not defined (because the denominator vanishes), and in a neighborhood of which the function therefore goes to $+\infty$ or $-\infty$. The partial fraction decomposition highlights the singularities of a rational function and gives more specific information about its behavior near a singularity.

Let us study how to carry out a partial fraction decomposition in practice:

Outline of method:

Step 1: We first need to achieve *proper* (aka 'bottom heavy') fractions, where the denominator has a higher degree than the numerator; improper fractions, i.e., rational functions whose numerator has a degree higher or equal to the denominator will be reduced by long division, splitting off a polynomial. We then deal with the remaining proper fraction.

Step 2: Now we have to factor the denominator. The following theorem guarantees that this is possible in principle:
Theorem: Any polynomial with real coefficients can be written as a product of linear and quadratic polynomials.
This may be difficult to carry out in practice, and if we get stuck here, then we are stuck for good. But if we can achieve this step in practice, all further steps are roputine. Factoring the denominator was the first $\int$implification of our example (1).

Let us pause a bit here: If you want to write a quadratic polynomial, e.g., x^2+4x+3 as a product of linear polynomials, you can always use the quadratic formula to find the zeros of that polynomial: $x^2+4x+3=0$ if and only if $x=-3$ or $x=-1$. This is how you find $x^2+4x+3=(x+1)(x+3)$. If the quadratic formula does not give any real zeros, as in the case of x^2+4x+5, you leave the quadratic polynomial alone. Zeros of the polynomial will always correspond to linear factors.

In the case (1), you have no feasible systematic way to find zeros of the denominator $x^4+x^3-x^2+x-2$. By guessing, you may however find that $x=1$ is a zero, and then you know

$$\begin{aligned} x^4+x^3-x^2+x-2 &= (x-1)(\text{poly' of deg 3, to be found by long division}) \\ &= (x-1)(x^3+2x^2+x+2) \end{aligned}$$

If you can guess another zero of the remaining factor (x^3+2x^2+x+2) –here, this would be $x=-2$–, you get $(x^3+2x^2+x+2)=(x+2)(x^2+1)$

In order to see how surprisingly strong this factorization theorem is, try the polynomial x^4+1. It has no zeros, so applying our theorem to it cannot produce linear factors. So, if our boldfaced theorem is true, it must be possible to write x^4+1 as a product of two quadratic polynomials. If you try to find how this will actually look: well, it will be quite sophisticated, you would probably not guess it. You have to find numbers p_1, q_1, p_2, q_2 such that $x^4+1=(x^2+p_1x+q_1)(x^2+p_2x+q_2)$. *Can you do this, by expanding the right hand side and comparing coefficients of powers of x? — In principle you can; but don't get disheartened: it takes some time to find the coefficients.* If you have actually carried it out, you'll be in for a surprise:[1]

We can now continue our itemized strategy of finding a partial fraction decomposition:

Step 3: **Theorem: Any proper fraction of polynomials can be decomposed into partial fractions according to the following example, which displays all features that could occur:** Given numbers $a_1, a_2, p_1, p_2, q_1, q_2$, and any polynomial in the numerator, numbers $b_1, b_2, b_3 \ldots$ can be found such that:

$$\frac{\text{any polyn' of degree less than the degree of the denominator}}{(x-a_1)(x-a_2)^4(x^2+p_1x+q_1)(x^2+p_2x+q_2)^2} =$$
$$= \frac{b_1}{(x-a_1)} + \frac{b_2}{(x-a_2)} + \frac{b_3}{(x-a_2)^2} + \frac{b_4}{(x-a_2)^3} + \frac{b_5}{(x-a_2)^4} +$$
$$+ \frac{b_6x+b_7}{x^2+p_1x+q_1} + \frac{b_8x+b_9}{x^2+p_2x+q_2} + \frac{b_{10}x+b_{11}}{(x^2+p_2x+q_2)^2}$$

In other words:

- For every nonrepeated linear factor in the denominator on the left (here $x-a_1$), you get one simple fraction (here, number$/(x-a_1)$) on the right.
- For every repeated linear factor in the denominator on the left (here $(x-a_2)^4$), you get as many simple fractions on the right as the number how often that factor was repeated, and their denominators echo the corresponding factor from the left, but with increasing powers from 1 up to what we had on the left.
- For every nonrepeated quadratic factor in the denominator on the left (here $x^2+p_1x+q_1$), you get one simple fraction with that very denominator on the right. The denominator of that fraction may be a linear polynomial now:
  $(b_6x+b_7)/(x^2+p_1x+q_1)$.
- For every repeated quadratic factor, you get similar fractions on the right, with increasing powers in the denominator.

[1] Answer: $x^4+1=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$. At least, check this by expanding.

What remains to be done is to see how you can actually find $b_1, b_2, \ldots$. This will be described in a moment. One of the procedures to find the b_i will actually contain all the ideas needed for a formal proof of the theorem.

Step 3a: The above partial fraction decomposition is actually not in optimal shape. It is preferrable, for theoretical as well as for calculational purposes, to write the quadratic polynomials in the denominator without linear term, by completing the squares. And it is also preferrable to group the numerator accordingly. So, for instance, a term like $\frac{3x+1}{x^2+4x+11}$ should be rewritten as $\frac{3(x+2)-5}{(x+2)^2+7}$. The rewriting of the denominator with completed squares makes its (complex) zeros more visible: in the given example, these would be $-2 \pm i\sqrt{7}$. From an integration point of view, we can evaluate $\int \frac{3(x+2)-5}{(x+2)^2+7}\,dx$ immediately: It is $\frac{3}{2}\ln[(x+2)^2+7] - \frac{5}{\sqrt{7}} \arctan \frac{x+2}{\sqrt{7}}$.

This ends the basic outline of partial fraction decomposition. We now discuss the way how you can actually find the numbers $b_1, b_2, b_3, \ldots$.

There is a simple-minded way that always works; however, in all but the simplest cases it will be unnecessarily tedious. But you should have understood it and tried for yourself, before you venture into the more sophisticated, but very fast way of doing it. For example, we need to find b_1, b_2, b_3 such that

$$\frac{x^2+2x+2}{x(x-1)(x+2)} = \frac{b_1}{x} + \frac{b_2}{x-1} + \frac{b_3}{x+2} \tag{2}$$

holds identically (for all x). So we bring the right hand side on a common denominator, and sort powers of x in the numerator:

$$\begin{aligned}
\frac{b_1}{x} + \frac{b_2}{x-1} + \frac{b_3}{x+2} &= \frac{b_1(x-1)(x+2)}{x(x-1)(x+2)} + \frac{b_2 x(x+2)}{x(x-1)(x+2)} + \frac{b_3 x(x-1)}{x(x-1)(x+2)} \\
&= \frac{b_1(x^2+x-2) + b_2(x^2+2x) + b_3(x^2-x)}{x(x-1)(x+2)} \\
&= \frac{x^2(b_1+b_2+b_3) + x(b_1+2b_2-b_3) + (-2b_1)}{x(x-1)(x+2)} \\
\text{and this should } &= \frac{x^2+2x+2}{x(x-1)(x+2)}
\end{aligned}$$

So comparing coefficients in the numerator, you need

$$\begin{array}{rcl}
b_1 + b_2 + b_3 = 1 & & \\
 & \oplus\!\!- & 2b_1 + 3b_2 = 3 \\
b_1 + 2b_2 - b_3 = 2 & & \\
-2b_1 = 2 & \text{———} & b_1 = -1
\end{array}$$

Therefore we get $b_1 = -1$, $b_2 = 5/3$, $b_3 = 1/3$.

This method is available in all cases, but it involves as many equations in as many unknowns as is specified by the degree of the denominator. — In contrast, here is a shorter method (called 'residue method' or 'cover-up method'), which will most easily apply to linear nonrepeated factors. With a slight modification for linear repeated factors, it will only give the coefficient of the highest power (in the big example given above, it would therefore only yield b_1 and b_5). Finally, the cover-up method will be available also with quadratic factors, provided you use complex numbers.

For simplicity, I will first assume we have nonrepeated linear factors only:

To determine b_1, b_2, b_3 *in turn* from (2), we multiply (2) by the corresponding denominators respectively, namely by x, $x-1$ and $x+2$. This is done in separate independent steps starting over from (2) each time. For b_1, multiplication by x transforms (2) into

$$\frac{x^2+2x+2}{(x-1)(x+2)} = b_1 + \left(\frac{b_2}{x-1} + \frac{b_3}{x+2}\right) \cdot x$$

We (pretend to) plug in $x=0$ into this equation, and get

$$\frac{0^2+2\cdot 0+2}{(0-1)(0+2)} = b_1 + (\ldots)\cdot 0$$

i.e., $-1 = b_1$ immediately. The method is called cover-up method, because it can be done without a lot of writing already from (2): To obtain b_1, you look at the left hand side, cover up exactly that term in the denominator that goes with b_1 on the right hand side, and then you plug in that number for x which would have made vanish the covered-up factor in the denominator.

I have been careful to say we *pretend to* plug in, rather than 'we plug in'. The reason is that $x=0$ is not legitimate to plug in into (2), exactly because of the vanishing denominator. Therfore, an equation derived from (2) is also not legitimate to be used for $x=0$. What we actually mean to do here is to calculate the *limit* as $x \to 0$. But the actual calculation of this limit will now (i.e., after having multiplied by x) amount practically to plugging in $x=0$.

There is kind of a philosophical message coming together with the partial fraction decomposition. You should consider those points x of a rational function f as its distinctive marks, where the denominator vanishes. If ever rational functions were wanted by the sheriff for wrongdoing, their vertical asymptotes would be the information given on the public announcement :-) With some embellishments added, there will be a result in advanced calculus to the effect that the behavior of a rational function near these points identifies that function nearly as uniquely as a fingerprint. To write a rational function in terms of partial fractions means to write it in such a way as to display certain of its essential features the most visibly. Displaying essential features as clearly as possible will simplify any scrutiny, in particular the search for an antiderivative. And the cover up method is so smart and efficient just because it uses those numbers for x where the essential things happen, namely where the denominator of the rational function vanishes. By focusing on the essential points (in the example $x=0$, $x=1$ and $x=-2$) – 'essential points' in the literal as well as in the figurative sense – we avoid unnecessary calculations and retreive b_1, b_2 and b_3 exactly at those places where they naturally belong.

Assessment and comparison of linear equation method vs. cover-up method

A-priori, there is one advantage of the 'tedious' method determining the coefficients by solving linear equations: It could be faithfully used even without a proof of the theorem that asserts that the PFD is indeed possible. In each case, when the linear equations turn out to have a solution, we *have shown* for this particular case, by our very calculation, that the PFD is possible. If, for instance, we had forgotten the long division from step 1 and attempted to write $\frac{x^3+x+2}{x^2-4x+3}$ as $\frac{a}{x-1} + \frac{b}{x-3}$, we would find out that the system of linear equations for a, b has no solution. In contrast, if we attempted the cover-up method, we would find $a = \frac{1^3+1+2}{1-3} = -2$ and $b = \frac{3^3+3+2}{3-1} = 16$, whereas in reality $\frac{x^3+x+2}{x^2-4x+3} = x+4-\frac{2}{x-1}+\frac{16}{x-3}$. The cover-up method would not have alerted us of the forgotten polynomial. It *assumes* that we can write the given rational in a certain form, and only a proof of the theorem that guarantees this possiblity makes the reasoning of the cover-up method logically sound.

However, for practical calculations, the advantage of the linear equations method turns into a disadvantage: We have to do a lot of work each time *because* we are not using the a-priori knowledge from the theorem. Moreover, any miscalculation of one coefficient in the system of linear equations will poison the rest of the calculation. In contrast, with the cover-up method, each coefficient is calculated independently of the others, and an error in one calculation will not poison the others.

Ramifications, and a Proof of the Method

Let's view the cover-up method from a different angle. We will now also allow repeated linear factors. We illustrate the method with the following example:

$$\frac{x^3+4x^2+2x-16}{(x+1)(x-2)^2(x-4)} = \frac{a}{x+1} + \frac{b}{(x-2)^2} + \frac{c}{x-2} + \frac{d}{x-4} \tag{3}$$

We will use a procedure that retains some benefits of the cover-up method, but also *proves* the validity of a decomposition of the form (3). First let's write (3) in an equivalent form by moving the first term on the right to the left side and putting it all on a common denominator:

$$\frac{x^3+4x^2+2x-16-a(x-2)^2(x-4)}{(x+1)(x-2)^2(x-4)} = \frac{b}{(x-2)^2} + \frac{c}{x-2} + \frac{d}{x-4} \tag{3'}$$

Eqn (3) can be solved *if and only if* eqn (3') can be solved. And there is only one way how (3') can possibly be true: Since the right hand side can be written with a common denominator $(x-2)^2(x-4)$ (not involving $x+1$ any more), the same must be true for the left hand side, and therefore the polynomial in the numerator must have a factor $(x+1)$ that could cancel the one in the denominator. But this means, the numerator must vanish if $x=-1$. In other words, the only way (3') can be possible is for a to equal $[(-1)^3+4(-1)^2+2(-1)-16]/[((-1)-2)^2((-1)-4)]$, which is exactly the value obtained by the cover-up method. It evaluates to $a = 1/3$.

Now conversely, algebra tells us: when a polynomial $p(x)$ vanishes at $x = x_0$, then we can actually factor it as $p(x) = (x-x_0)q(x)$ with a polynomial q of degree one lower than that of p. So we know now that we can indeed cancel $(x+1)$ on the left hand side. We calculate

$$x^3+4x^2+2x-16-\frac{1}{3}(x-2)^2(x-4) = \frac{2}{3}x^3+\frac{20}{3}x^2-\frac{14}{3}x-\frac{32}{3} = \frac{2}{3}(x+1)(x^2+9x-16) \tag{$*$}$$

and conclude that (3) and (3') have solutions if and only if $a = \frac{1}{3}$ and the reduced problem

$$\frac{\frac{2}{3}(x^2+9x-16)}{(x-2)^2(x-4)} = \frac{b}{(x-2)^2} + \frac{c}{x-2} + \frac{d}{x-4} \tag{3''}$$

can be solved. Note also that you would not have *guessed* the factorization $(*)$ if you hadn't *known* from the preceding calculation that $x+1$ must be a factor.

We started with a bottom-heavy fraction in (3), and now in (3") we still have a bottom-heavy fraction, because the degrees of both numerator and denominator were reduced by one. This observation would become an induction step in a formal proof of the PFD decomposition theorem. We can prove the PFD theorem for bottom-heavy rational functions of degree $k+1$ (in our example $k+1=4$) under the induction hypothesis that it is proved already for bottom-heavy rational functions of degree k.

Let's do another reduction step to show that the presence of repeated linear factors is not a problem after all; the crucial step is that with repeated factors, we determine the coefficient for the highest power first. We want to obtain b:

$$\frac{\frac{2}{3}(x^2+9x-16)-b(x-4)}{(x-2)^2(x-4)} = \frac{c}{x-2} + \frac{d}{x-4} \tag{3'''}$$

So we need $b = \frac{2}{3}(2^2+9\cdot 2-16)/(2-4)$ (again by the cover-up method) in order to cancel one factor $(x-2)$ from the fraction on the left, with the same reasoning as before. We evaluate this to $b=-2$, and now we refactor the numerator on the left and cancel again:

$$\frac{2}{3}(x^2+9x-16)+2(x-4) = \frac{2}{3}x^2+8x-\frac{56}{3} = \frac{2}{3}(x-2)(x+14)$$

and

$$\frac{\frac{2}{3}(x+14)}{(x-2)(x-4)} = \frac{c}{x-2} + \frac{d}{x-4} \tag{3''''}$$

Next we would get $c=-16/3$ and $d=6$ by the same step by step reduction.

Except for the somewhat informal exposition with the induction given only implicitly, this proves the PFD theorem in the case where the denominator has only linear factors.

In practical calculations, we would rather use the cover-up method; but in cases with repeated factors, it only gives us the coefficient of the highest power. We do need to carry out a reduction step to get coefficients for the smaller powers of repeated factors. Note that these reduction steps are automatically error-detecting: If a miscalculation has occurred, then the term that we know should factor off and cancel will not factor off, and this detects that a miscalculation must have occurred.

Quadratic terms in the denominator

Quadratic terms in the denominator are merely an artefact of the fact that we want to work with real numbers instead of complex numbers.

Let us review some facts about polynomials that we have been abd will be relying on:

Fundamental Theorem of Algebra: *Every polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_1 x + a_0$ of degree n with complex coefficients a_j (and $a_n \neq 0$) can be written as a product $p(x) = a_n(x-x_1)\cdots(x-x_n)$ with appropriate complex numers x_j. If the coefficients a_j are real numbers, then those of the x_j that are* not *real numbers come in pairs of complex conjugate numbers $a+bi$ and $a-bi$, with real a,b ($b\neq 0$). Then the two linear factors $(x-(a+bi))(x-(a-bi))$ can be combined to a quadratic factor with real coefficients $(x-a)^2+b^2$.*

Note: two complex numbers are complex conjugate if they have the same real part and opposite imaginary parts, e.g., the complex conjugate to $4+7i$ is $4-7i$ and vice versa.

The algorithm from the previous section can be carried out with complex numbers just as well. It is not difficult to see (but we're not bothering about details) that the coefficients for complex conjugate partial fractions are themselves, complex conjugate, in other words: if we have found coefficients c and c' that contribute to $\frac{c}{x-a-bi} + \frac{c'}{x-a+bi}$, then c and c' are complex conjugates. (Similarly for higher powers $\frac{c}{(x-a-bi)^k} + \frac{c'}{(x-a+bi)^k}$.) Combining these complex conjugate pairs and slightly rearranging, we obtain the claimed partial fraction decomposition in full generality.

With this background knowledge, we can even use the cover-up method to determine partial fraction coefficients in the presence of quadratic factors: For instance, take

$$\frac{x^3+2x+97}{(x^2+2x+2)(x^2-4x+13)} = \frac{a(x+1)+b}{(x+1)^2+1} + \frac{c(x-2)+d}{(x-2)^2+9} \tag{4}$$

and note in particular how I have organized the linear terms in the numerators on the right to align with the structure of their denominator, as mentioned in Step 3a of the 'Outline'.

Short of the use of complex numbers, there is probably no way to avoid solving a full system of four equations in four unknowns. However, with complex numbers, we can simply multiply (4) by $(x+1)^2+1$ and the (pretend to) plug in $x=-1+i$. (More precisely, we consider the limit $x\to -1+i$. Let me mention that the definitions and elementary theorems about limits carry over to complex numbers immediately.) Just as well, we could have chosen $x\to -1-i$, one of the pair of complex conjugates is as good as the other. The key is that $(x+1)^2+1$ vanishes for $x=-1\pm i$. We get

$$\frac{x^3+2x+97}{(x^2-4x+13)} = a(x+1)+b+\frac{c(x-2)+d}{(x-2)^2+9}\,((x+1)^2+1) \tag{4'}$$

and then

$$\frac{(-1+i)^3+2(-1+i)+97}{(-1+i)^2-4(-1+i)+13} = a((-1+i)+1)+b+\frac{\cdots}{\cdots}\times 0 \tag{4''}$$

This takes a bit of arithmetic, but we conclude $5+2i=ai+b$, and therefore $a=2$, $b=5$.

Similarly, we calculate the other two coefficients $c=-1$, $d=1$ from

$$\frac{(2+3i)^3+2(2+3i)+97}{(2+3i)^2+2(2+3i)+2} = c\,3i+d$$

Caveat

There is no good reason to discriminate against complex numbers as far as basic arithmetic and the limit notion is concerned. In the chapter on Taylor series, we will see, that, as a consequence of this, it is perfectly natural to consider exponentials, trigonometric and hyperbolic functions of a complex argument. Inasmuch as you handle derivatives by means of the usual calculation rules (chain rule, product, sum, quotient rules), these also carry over to complex variables.

What you forfeit is the convenience of drawing function graphs (and this is probably the primary reason why calculus textbooks discriminate against complex numbers.)

There is however one issue where you want to shun complex variables until such time when you have a formal training in them, because otherwise you'd run into inconsistencies that you could not understand or resolve. You do *not* want to write down roots, logarithms, inverse trigs and inverse hyperbolics of complex arguments. The reason for this prohibition is simple, and can be understood easiest in terms of the square root: $\sqrt{2}$ is defined to be the *positive* real number whose square is 2. We had to restrict the domain of the function $f(x)=x^2$ somehow to make it one-to-one and be able to invert it. As long as we are within the real numbers, restricting x^2 to $[0,\infty[$ (and therefore defining $\sqrt{y}$ as the *positive* square root) was a perfectly good choice. But with complex numbers, many restrictions are possible, and none is really good for all purposes. So, an expression like $\sqrt{2+3i}$ must be deemed as *undefined*, unless you precede it by a defining sentence or two (which you would not be able to write competently at this stage). If you recklessly pretend otherwise, you may end up in contradictions.

This means, you can well use complex numbers for the *algebra* of finding the PFD. Once you have done this and recombined complex conjugates to form real terms, *then* (and no sooner) do you do the integrals, which will then involve logarithms and arctangents of *real* quantities.

Let me be throw in another diatribe: The fact that you wouldn't handle gasoline when you have open fire around is not deemed to be a reason against driving a car that runs on gasoline. Likewise, the fact that you need to leave complex numbers away in certain situations should not be a reason to discriminate against them in other situations where they are very useful and perfectly benign.

Some practice problems

Here are a few examples for practising:

$(a)\qquad \displaystyle\int_0^1 \frac{dx}{x^3+1}$

$(b)\qquad \displaystyle\int_2^4 \frac{x^2+2x+3}{(x-1)^2(x^2+1)}\,dx$

$(c)\qquad \displaystyle\int_0^1 \frac{x^4+1}{x^2+1}\,dx$

$(d)\qquad \displaystyle\int_0^1 \frac{dx}{x^4+1}$ this is a tough one, because of the difficulty of factoring

$(e)\qquad \displaystyle\int_0^1 \frac{x\,dx}{x^4+1}$ this one is much easier – simplify the integral by a substitution first!

$(f)\qquad \displaystyle\int_0^1 \frac{x+1}{x^2+x+1}\,dx$

Error detection

Whichever method you use, it's nice to have an easy error detection. Take (3) for example. We had found that

$$\frac{x^3+4x^2+2x-16}{(x+1)(x-2)^2(x-4)} = \frac{1/3}{x+1} + \frac{-2}{(x-2)^2} + \frac{-16/3}{x-2} + \frac{6}{x-4}$$

Since the claim is that this is true for all x, we can use any x not used so far as an independent error checking method: For instance we could plug in $x=0$ or $x=1$. One smart method is to study the limit $x\to\infty$ wisely: Of course both sides go to 0 as $x\to\infty$, regardless of the values of a,b,c,d, because we are having bottom heavy fractions. So this naive way doesn't check against miscalculations. However, if you multiply equation (3) by x, and then take the limit $x\to\infty$, you get 1 on the left hand side, and $a+c+d$ on the right hand side. So we are checking whether indeed $\frac{1}{3}-\frac{16}{3}+6=1$.

Solution to practice problems

Not all intermediate steps are carried out, but essential steps are given.

$\boxed{\text{(a)}}$

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1}$$

with $a = \frac{1}{3}$ (cover up method). $b = -\frac{1}{3}$, $c = \frac{2}{3}$ can be found either by reduction (move $a/(x+1)$ term over, simplify and cancel), or else by complex cover up $x \to \frac{1}{2} + \frac{\sqrt{3}i}{2}$. (solve linear equations for unknown coefficients). Must complete square in denominator and separate fractions in order to reduce to standard integrals:

$$\frac{1}{3}\left(\frac{-x+2}{x^2-x+1}\right) = \frac{1}{3}\left(\frac{-x+2}{(x-\frac{1}{2})^2+\frac{3}{4}}\right) = \frac{1}{3}\left(\frac{-(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\right)$$

$$\begin{aligned}
\int_0^1 \frac{dx}{x^3+1} &= \frac{1}{3}\int_0^1 \left(\frac{1}{x+1} - \frac{(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\right) dx = \\
&= \frac{1}{3}\left([\ln(x+1)]_0^1 - \left[\frac{1}{2}\ln\left((x-\frac{1}{2})^2+\frac{3}{4}\right)\right]_0^1 + \left[\frac{3}{2}\frac{2}{\sqrt{3}}\arctan\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right]_0^1\right) \\
&= \frac{1}{3}\left(\ln 2 + 0 + \sqrt{3}\arctan(\frac{1}{\sqrt{3}}) - \arctan(-\frac{1}{\sqrt{3}})\right) = \frac{1}{3}\left(\ln 2 + \frac{\pi}{3}\sqrt{3}\right)
\end{aligned}$$

$\boxed{\text{(b)}}$

$$\begin{aligned}
\int_2^4 \frac{x^2+2x+3}{(x-1)^2(x^2+1)}\,dx &= \int_2^4 \left(\frac{3}{(x-1)^2} - \frac{1}{(x-1)} + \frac{x-1}{x^2+1}\right) dx = \\
&= \left[\frac{-3}{(x-1)} - \ln|x-1| + \frac{1}{2}\ln(x^2+1) - \arctan x\right]_2^4 = 2 + \arctan 2 - \arctan 4 + \frac{1}{2}\ln\frac{17}{45}
\end{aligned}$$

$\boxed{\text{(c)}}$

$$\int_0^1 \frac{x^4+1}{x^2+1}\,dx = \int_0^1 \left(x^2 - 1 + \frac{2}{x^2+1}\right) dx = \frac{1}{3} - 1 + 2\arctan 1 = \frac{\pi}{2} - \frac{2}{3}$$

$\boxed{\text{(d)}}$

Factorization of denominator: see page 2.

$$\frac{1}{x^4+1} = \frac{ax+b}{x^2+\sqrt{2}x+1} + \frac{cx+d}{x^2-\sqrt{2}x+1}$$

With the inconvenient numbers here, we may just try to solve the equations for a, b, c, d:

$$\begin{array}{ll}
x^0: & b+d=1 \\
x^1: & a+c+(d-b)\sqrt{2}=0 \\
x^2: & b+d+(c-a)\sqrt{2}=0 \\
x^3: & a+c=0
\end{array}
\qquad
\begin{array}{l}
\text{“}x^3\text{”\&“}x^1\text{”:} \qquad d=b \\
\Longrightarrow \text{(with “}x^0\text{”)} \quad b=d=\frac{1}{2} \\
\Longrightarrow a=\sqrt{2}/4=-c
\end{array}$$

Alternatively, we could have used the complex cover-up method at $x \to \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$ and $x \to -\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$ respectively.

$$\int_0^1 \frac{(\sqrt{2}/4)x+1/2}{x^2+\sqrt{2}x+1}\,dx = \frac{\sqrt{2}}{4}\int_0^1 \left(\frac{x+\sqrt{2}/2}{(x+\sqrt{2}/2)^2+1/2} + \frac{\sqrt{2}/2}{(x+\sqrt{2}/2)^2+1/2}\right)dx =$$
$$= \frac{\sqrt{2}}{8}\left[\ln\Big((x+\sqrt{2}/2)^2+1/2\Big)\right]_0^1 + \frac{\sqrt{2}}{4}\left[\arctan\frac{x+\sqrt{2}/2}{\sqrt{2}/2}\right]_0^1 =$$
$$= \frac{\sqrt{2}}{8}\ln(2+\sqrt{2}) + \frac{\sqrt{2}}{4}\Big(\arctan(1+\sqrt{2}) - \frac{\pi}{4}\Big)$$

Similarly,

$$\int_0^1 \frac{(-\sqrt{2}/4)x+1/2}{x^2-\sqrt{2}x+1}\,dx = -\frac{\sqrt{2}}{4}\int_0^1 \left(\frac{x-\sqrt{2}/2}{(x-\sqrt{2}/2)^2+1/2} - \frac{\sqrt{2}/2}{(x-\sqrt{2}/2)^2+1/2}\right)dx =$$
$$= -\frac{\sqrt{2}}{8}\left[\ln\Big((x-\sqrt{2}/2)^2+1/2\Big)\right]_0^1 + \frac{\sqrt{2}}{4}\left[\arctan\frac{x-\sqrt{2}/2}{\sqrt{2}/2}\right]_0^1 =$$
$$= -\frac{\sqrt{2}}{8}\ln(2-\sqrt{2}) + \frac{\sqrt{2}}{4}\Big(\arctan(-1+\sqrt{2}) + \frac{\pi}{4}\Big)$$

Now $\ln(2+\sqrt{2}) - \ln(2-\sqrt{2}) = \ln\frac{(2+\sqrt{2})}{(2-\sqrt{2})} = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1} = \ln((\sqrt{2}+1)^2)$. You are certainly not expected to know that $\arctan(\sqrt{2}+1) = 3\pi/8$, $\arctan(\sqrt{2}-1) = \pi/8$, but these are true, and so the final result can be simplified to

$$\int_0^1 \frac{dx}{x^4+1} = \frac{\sqrt{2}}{4}\Big\{\ln(\sqrt{2}+1) + \pi/2\Big\}$$

$\boxed{\text{(e)}}$

$$\text{subst. } u = x^2: \quad \int_0^1 \frac{x\,dx}{x^4+1} = \left[\frac{1}{2}\arctan(x^2)\right]_0^1 = \frac{\arctan 1 - \arctan 0}{2} = \frac{\pi}{8}$$

$\boxed{\text{(f)}}$

$$\int_0^1 \frac{x+1}{x^2+x+1}\,dx = \int_0^1 \left(\frac{x+1/2}{(x+1/2)^2+3/4} + \frac{1/2}{(x+1/2)^2+3/4}\right)dx = \frac{1}{2}\ln 3 + \frac{\pi}{6\sqrt{3}}$$