Math 341 Homework 7

1. (Composition of continuous functions) Suppose A and B are subsets of $\mathbb{R}$, $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are functions, and $f(A) \subseteq B$. Suppose f is continuous at $c \in A$ and g is continuous at $f(c)$. Prove that the compostion $g \circ f$ is continuous at c.

2. Let $K \subseteq \mathbb{R}$ be nonempty and compact. Suppose $f : K \to \mathbb{R}$ is continuous. Prove that there exists $x_1 \in K$ such that $f(x_1) = \inf\{f(x) : x \in K\}$.

3. Suppose $O \subseteq \mathbb{R}$ is open and $f : O \to \mathbb{R}$ is a function. Suppose that for all $U \subseteq \mathbb{R}$ such that U is open, we have $f^{-1}(U) = \{x \in O : f(x) \in U\}$ is open. Prove that f is continuous on O.

4. Suppose $E \subseteq \mathbb{R}$ is closed. Suppose $f : E \to \mathbb{R}$ is continuous. Show that $f^{-1}(F) = \{x \in E : f(x) \in F\}$ is closed, for every closed set $F \subseteq \mathbb{R}$.

5. Suppose K is compact, and $f : K \to \mathbb{R}$ is continuous. Prove that $f(K)$ is compact. Hint: use sequential compactness.

6. Show that if f is increasing on $[a, b]$ and f has the intermediate value property, then f is continuous on $[a, b]$.

7. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.

   (a) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

   (b) Show that for each $n \in \mathbb{N}$ there exist $x_n, y_n \in [0, 1]$ satisfying $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.