Math 341 Homework 4

1. Let (a_n) be a bounded sequence. Define the sequence (y_n) by $y_n = \sup\{a_k : k \geq n\}$. Prove that the sequence (y_n) converges.

2. Suppose $C > 0$. Define a function $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \dfrac{1}{2}\left(x + \dfrac{C}{x}\right)$.

   (a) Prove that $f(x) \geq \sqrt{C}$ for any $x > 0$. Hint: start from $(x - \sqrt{C})^2 \geq 0$.

   (b) Prove that if $x \geq \sqrt{C}$, then $f(x) \leq x$.

   (c) Define a sequence x_n inductively as follows: let x_1 be any real number satisfying $x_1 > \sqrt{C}$. For $n \geq 1$, define $x_{n+1} = f(x_n)$. Show that x_n converges to $\sqrt{C}$. Remark: This is the algorithm for computing $\sqrt{C}$ that we discussed on the first day of class.

3. (a) Suppose $\sum_{k=1}^{\infty} a_k$ is a convergent series and $c \in \mathbb{R}$. Prove that $\sum_{k=1}^{\infty}(ca_k)$ converges and

$$\sum_{k=1}^{\infty}(ca_k) = c\sum_{k=1}^{\infty} a_k.$$

   (b) Suppose $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series. Prove that $\sum_{k=1}^{\infty}(a_k + b_k)$ is convergent and

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

4. (a) Suppose $r \in \mathbb{R}, r \neq 1$, and $n \in \mathbb{N}$. Prove that $\displaystyle\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$.

   (b) (Geometric series) Suppose $|r| < 1$. Prove that the geometric series $\sum_{k=0}^{\infty} r^k$ converges and $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$.

   (c) Consider the infinite decimal $.\overline{9} = .999999\ldots$, that is, $.a_1a_2a_3\cdots a_k\cdots$ where $a_k = 9$ for all $k \in \mathbb{N}$. Interpret $.\overline{9}$ as an infinite series and use part (b) to evaluate the series.

5. (Absolute convergence implies convergence) Suppose $\sum_{k=1}^{\infty} |a_k|$ converges. Prove that $\sum_{k=1}^{\infty} a_k$ converges.

   Hint: write $a_k = |a_k| + a_k - |a_k|$, show that $\sum_{k=1}^{\infty}(|a_k| + a_k)$ is a series of nonnegative terms, show it converges by the comparison test, and then apply Problem 3.

6. (Alternating series test) Suppose a_k is a sequcnce of positive real numbers which is strictly decreasing: $a_{k+1} < a_k$ for all $k \in \mathbb{N}$. Suppose also that $\lim_{k\to\infty} a_k = 0$. Consider the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$. Let $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ be the nth partial sum of the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$.

   (a) Prove that the subsequence $s_{2n} = (s_2, s_4, s_6, \ldots)$ is strictly increasing: $s_{2n+2} > s_{2n}$ for all $n \in \mathbb{N}$.

   (b) Prove that the subsequence $s_{2n-1} = (s_1, s_3, s_5, \ldots)$ is strictly decreasing: $s_{2n+1} < s_{2n-1}$ for all $n \in \mathbb{N}$.

   (c) Prove that for all $k, j \in \mathbb{N}$, we have $s_{2j} < s_{2k-1}$.

   (d) Deduce that $\lim_{n\to\infty} s_{2n}$ and $\lim_{n\to\infty} s_{2n-1}$ exist and prove that these two limits are equal.

   (e) Prove that $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.