Math 341 Homework 3

1. Let A be an infinite set. Prove that if $A \subseteq X$, and X can be written as $X = \bigcup_{k=1}^{n} X_k$, then at least one of the sets $X_k \cap A$ is an infinite set.

2. Prove the following characterization of convergence. If (x_n) is a convergent sequence, then for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ so that whenever $m, n \geq N$, it follows that $|x_m - x_n| < \varepsilon$.

3. Find a recursive formula for the sequence

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

   Show that the sequence is increasing and bounded above by 4.

4. (a) Prove that for any $x, y \in \mathbb{R}$, we have

$$|\,|x| - |y|\,| \leq |x - y|.$$

   (b) Suppose s_n is a sequence of real numbers, $\ell \in \mathbb{R}$, and $\lim_{n\to\infty} s_n = \ell$. Prove that $\lim_{n\to\infty} |s_n| = |\ell|$.

5. (a) Let $x \in \mathbb{R}$. Prove that $x = 0$ if and only if: for all $\epsilon > 0$, we have $|x| < \epsilon$.

   (b) (Uniqueness of limits) Suppose s_n is a sequence of real numbers, $\ell_1, \ell_2 \in \mathbb{R}$, $\lim_{n\to\infty} s_n = \ell_1$, and $\lim_{n\to\infty} s_n = \ell_2$. Prove that $\ell_1 = \ell_2$.

6. Let s_n be a sequence of real numbers and let $\ell \in \mathbb{R}$. Prove that the following 3 statements are equivalent (that is, any one of the statements implies any other one of the statements):

   i. $\lim_{n\to\infty} s_n = \ell$;

   ii. $\lim_{n\to\infty}(s_n - \ell) = 0$;

   iii. $\lim_{n\to\infty} |s_n - \ell| = 0$.

7. Suppose s_n and t_n are sequences of real numbers, s_n is a bounded sequence, and $\lim_{n\to\infty} t_n = 0$. Prove that $\lim_{n\to\infty}(s_n t_n) = 0$.

8. Prove that if a sequence (a_n) is decreasing and bounded below, then $\lim a_n = \inf\{a_n : n \in \mathbb{N}\}$.