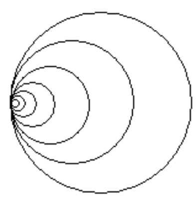
Homotopy Critical Values of Metric Spaces

In 2009 the HC group made great progress working homotopy critical values, and the 2010 project will continue in new directions. A metric space is simply a set with a "distance function" d that assigns to each pair of points a non-negative real number $d(x, y)$, interpreted as the "distance" between the points. This function must also be positive definite ($d(x, y) = 0$ if and only if $x = y$), symmetric ($d(x, y) = d(y, x)$) and satisfy the triangle inequality ($d(x, y) \leq d(x, z) + d(z, y)$)) for all $x, y, z \in X$. For $\varepsilon > 0$ there is a basic construction of a "metric covering space" X_ε with a function $\phi_\varepsilon : X_\varepsilon \to X$ that involves only elementary concepts. The space X_ε in some sense "unrolls" X, eliminating "holes" of size 3ε. Suppose, for example, C is the unit circle in the plane, with the intrinsic metric. That is, the distance between two points $\mathbf{x}$, $\mathbf{y}$ is not the distance between them as measured in the plane, but as though it were the distance a creature confined to the circle would have to travel to go from $\mathbf{x}$ to $\mathbf{y}$. In other words, $d(\mathbf{x}, \mathbf{y})$ is the length of the shortest path joining $\mathbf{x}$ and $\mathbf{y}$ that stays in C. For example, if $\mathbf{x}$ and $\mathbf{y}$ are antipodal points, $d(\mathbf{x}, \mathbf{y}) = \pi$, rather than the Euclidean distance 2. Now if $\varepsilon \geq \frac{\pi}{3}$, C_ε is just an exact copy of C, and ϕ_ε is essentially the identity map. But when $\varepsilon < \frac{\pi}{3}$, C_ε is the real line $\mathbb{R}$, i.e. an "unrolled" circle. The function $\phi_\varepsilon : \mathbb{R} \to C$ is essentially the one defined by $\phi_\varepsilon(t) = (\cos t, \sin t)$, and is known as the universal covering map of C. More complicated spaces have "holes" of different sizes, possibly even infinitely many holes of arbitrarily small size, as is the case with the so-called Hawaian Earring H pictured here.



H unrolls in discrete stages, and the values ε for which more of H unrolls are called *homotopy critical values*. Homotopy critical values are closely related to the *covering spectrum* studied by Sormani and Wei ([2]), which in turn is related to important questions involving Gromov-Hausdorff convergence. For this project students will work on analogous theorems concerning homotopy critical values. The only background required is at least one course involving rigorous proofs.

References

[1] V. Berestovskiǐ and C. Plaut, Uniform universal covers of uniform spaces, Top. Appl. 154 (2007) 1748-1777.

[2] C. Sormani and G. Wei, The covering spectrum of a compact length space, Journal of Differential Geometry 66 (2004) 647-689.