1. Let $H$ be a symmetric $n \times n$ matrix and $g$ a non-zero vector. Suppose you attempt to solve $Hx = -g$ using the linear conjugate gradient method (Algorithm 5.2) with $x_0 = 0$. (Assume all arithmetic is exact)

   (a) Show that if $H$ is positive definite then $x_n$ is a descent direction, i.e. $x_n^T g < 0$.

   (b) Show that every $x_k$ for $1 \leq k \leq n$ is a descent direction if $H$ is positive definite.

   (c) If $H$ is not positive definite, what quantities that can be computed during the process can be used to determine when $x_k$ is not a descent direction?

2. Let $f(x) = 10x_1^2 + x_2^2 - x_1x_2$ which has minimizer $x^* = (0, 0)^T$. With $x_0 = (1, 0)^T$ and an exact step-length search, compute $B_1$

   (a) using BFGS, with $B_0 = I$.

   (b) using BFGS, with $B_0$ a diagonal matrix with entries 20 and 2.

   (c) using BFGS, with $B_0 = \nabla^2 f(x_0)$.

   (d) using SR1, with $B_0 = I$.

3. Verify that the 4 updates (PSB, BFGS, DFP, SR1) all produce $B_{k+1}$ which satisfy the secant equation: $B_{k+1}s_k = y_k$.

4. Write out an efficient algorithm for performing the BFGS update for the approximate inverse Hessian ($H_{k+1}$), equation (8.16). Estimate the ‘cost’ of the algorithm, by assigning a cost of $2n^2$ to every matrix-vector product, $2n$ to every dot product and $n$ to every vector add and every scalar-vector product. You should be able to avoid any matrix-matrix products, as they would have a cost of $2n^3$.

5. Use the Sherman-Morrison-Woodbury Formula to find the inverse of

   $$A + ab^T + cd^T.$$