**Rate of Convergence (Q-Convergence)**

Let \( \{x^k\} \) be any infinite sequence. Let \( s^k = \max_{l \geq k} x^l \) and define

\[
\limsup_{k \to \infty} x^k = \lim_{k \to \infty} s^k.
\]

Sometimes \( \limsup \) is written as \( \lim \sup \). A \( \limsup \) always exists (if we allow \( +\infty \)). If \( \lim x^k = L \) then \( \limsup x^k = L \), but the opposite is not true.

**Examples:**

1. \( x^k = (-1)^k \), then \( s^k = 1 \) and \( \limsup x^k = 1 \).
2. \( x^k = \sin(k) \) then \( s^k = 1 \) and \( \limsup x^k = 1 \) (note in this case there is no obvious convergence).

Assume \( \lim_{k \to \infty} x^k = \hat{x} \) and there is some \( M \) such that \( x^k \neq \hat{x} \) for all \( k > M \). Then for \( p \geq 0 \) let

\[
C(p) = \limsup_{k \to \infty} \frac{|x^{k+1} - \hat{x}|}{|x^k - \hat{x}|^p}.
\]

Then, if \( C(p^*) < \infty \) for some \( p^* \) then \( C(p) = 0 \) for \( p < p^* \). If \( C(p^*) > 0 \) for some \( p^* \) then \( C(p) = \infty \) for \( p > p^* \). Both of these results come from the equality

\[
\frac{|x^{k+1} - \hat{x}|}{|x^k - \hat{x}|^p} = \frac{|x^{k+1} - \hat{x}|}{|x^k - \hat{x}|^{p^*}} |x^k - \hat{x}|^{p^* - p}.
\]

So, there exists a \( p^* \) (possibly infinite) such that

\[
C(p) = \begin{cases} 
0 & \text{if } 0 \leq p < p^* \\
C(p^*) & \text{if } p = p^* \\
\infty & \text{if } p > p^* 
\end{cases}
\]

This number \( p^* \) is the order of convergence for the sequence \( x^k \) and determines the rate of convergence as follows:

- If \( p^* = 1 \) and \( C(1) = 1 \) then we say the convergence is sublinear.
- If \( p^* = 1 \) and \( 1 > C(1) > 0 \) then we say the convergence is linear.
- If \( p^* > 1 \) or \( C(1) = 0 \) then we say the convergence is superlinear.
- If \( p^* = 2 \) then we say the convergence in quadratic.
- If \( p^* = 3 \), convergence is cubic, etc.

When working with convergence estimates it is often useful to use the following approximation:

\[
|x^{k+1} - \hat{x}| \approx C|x^k - \hat{x}|^{p^*}
\]

for some constant \( C \) (not necc. \( C(p^*) \)).