Multivariable Calculus and Analysis Review

- Sequences
  - Infinite Sequence: $\{x_k\}$ where $x_k \in \Re^n$
  - Convergence: $\lim_{k\to\infty} x_k = x$ if $\lim_{k\to\infty} \|x - x_k\| = 0$ for any norm; we also write $x_k \to x$
    This also implies term by term convergence
    Rate of convergence discussed in numerics review
- Sets: Typically we'll use sets $\Omega \subset \Re^n$ which are either open (doesn't include boundary) or closed (include boundary).
  Ω is *convex* if for all $x, y \in \Omega$, $tx + (1-t)y \in \Omega$ for all $0 \le t \le 1$
- Functions
  - Domains and Types: Typically functions will take values from $\Re^n$ and produce real values, i.e. $f : \Omega \to \Re$ where $\Omega \subset \Re^n$
    Can also have $f : \Re^n \to \Re^m$, etc.
    f is continuous at x if for any sequence $\{x_k\}$ with $x_k \to x$ and $x_k \neq x$, $f(x_k) \to f(x)$.
    f is *Lipschitz* with Lipschitz constant γ on Ω if for all $x, y \in \Omega$, $\|f(x) - f(y)\| \le \gamma \|x - y\|$
    Lipschitz implies continuous
    f is *convex* if Ω is convex and if $x, y \in \Omega$ implies $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$ for $0 \le t \le 1$; if it holds with $\ge$ f is *concave*.
  - Derivatives: For general $f : \Re^n \to \Re^m$, the derivatives of f involve the partial derivatives

$$\frac{\partial (f)_j}{\partial x_i}, \qquad 1 \le i \le n, \quad 1 \le j \le m$$

    For $f : \Re^n \to \Re$ $(m = 1)$, we have
    $(\nabla f(x))_i = \partial f / \partial x_i$ (gradient of f)
    $(\nabla^2 f(x)) = \partial^2 f / \partial x_i \partial x_j$ (Hessian of f)
    f is *continuously differentiable* if ∇f exists and is continuous
    f is *twice continuously differentiable* if $\nabla^2 f$ exists and is continuous; in this case $\nabla^2 f$ is symmetric
    If $p \in \Re^n$, $p \neq 0$, then the *directional derivative* of f in the direction p is given by $\lim_{\epsilon\to 0}(f(x + \epsilon p) - f(x))/\epsilon$ and is equal to $\nabla f(x)^T p$ (when f is cont. diff.)
- Theorems (like Mean Value & Taylor)
  - If f is cont. diff. and $0 \neq p \in \Re^n$, then $f(x+p) = f(x) + \nabla f(z)^T p$ for some $z = x + tp$, $0 < t < 1$
  - If f is twice cont. diff. and $0 \neq p \in \Re^n$, then $f(x+p) = f(x) + \nabla f(x)^T p + \frac{1}{2} p^T \nabla^2 f(z) p$ for some z as above
  - If f is twice cont. diff., $0 \neq p \in \Re^n$ and $\nabla^2 f$ is Lipschitz, then $|f(x+p) - f(x) - \nabla f(x)^T p - \frac{1}{2} p^T \nabla^2 f(x) p| \le \frac{\gamma}{6} \|p\|^3$