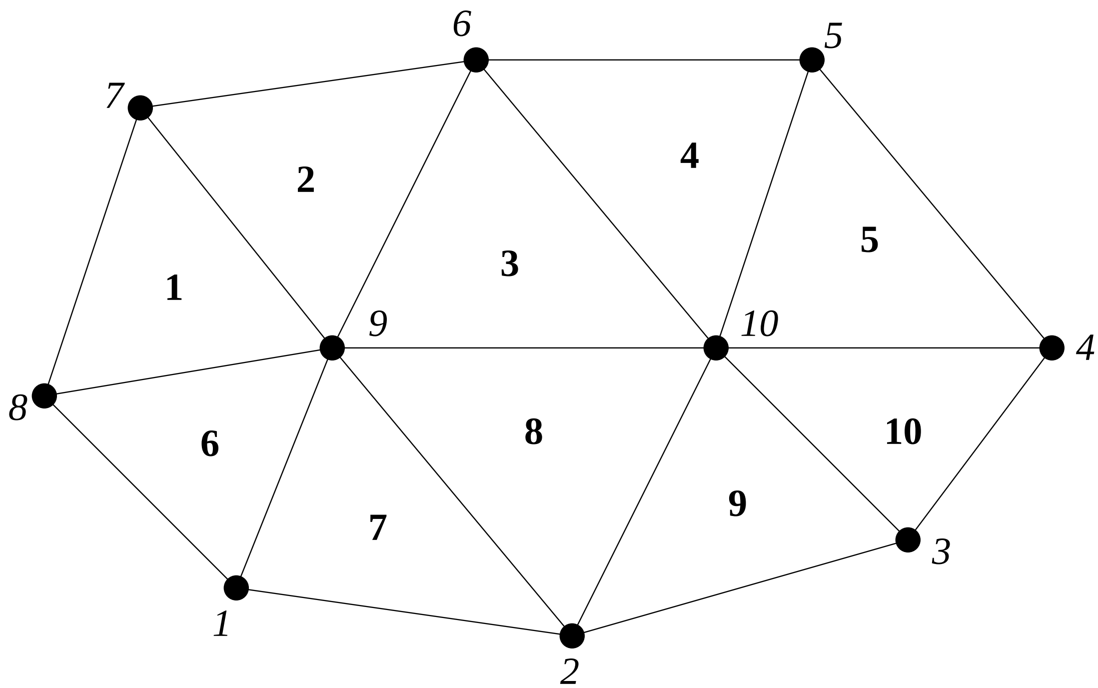
Homework #4 Solutions

Exercise 5: Matrix Assembly

Obviously the answers will vary depending on what triangulation you draw and how you number the vertices and the triangles. I provide here just an example.



The ten triangles are numbered in **bold** and the ten vertices are numbered in *italics*. The array defining the triangles is given by

$$T = \left[\begin{array}{cccccccccc} 9 & 9 & 10 & 10 & 4 & 1 & 2 & 2 & 3 & 3 \\ 7 & 6 & 6 & 5 & 5 & 9 & 9 & 10 & 10 & 4 \\ 8 & 7 & 9 & 6 & 10 & 8 & 1 & 9 & 2 & 10 \end{array} \right].$$

Assuming $M(k,i,j)$ is the value computed on triangle k with basis functions (locally numbered) i and j, we have (as a few examples):

$$\begin{array}{rcl} A(9,9) & = & M(3,3,3) + M(2,1,1) + M(1,1,1) + M(6,2,2) + M(7,2,2) + M(8,3,3) \\ A(10,10) & = & M(5,3,3) + M(4,1,1) + M(3,1,1) + M(8,2,2) + M(9,2,2) + M(10,3,3) \\ A(9,10) & = & M(3,3,1) + M(8,3,2) \end{array}$$

Exercise 6: Reference Element Calculations

Let $\hat{T}$ be the reference triangle with vertices $\hat{z}_1 = (0,0)$, $\hat{z}_2 = (1,0)$ and $\hat{z}_3 = (0,1)$. Let T be another triangle with vertices $z_1 = (0.2, 0.3)$, $z_2 = (0.1, 0.4)$ and $z_3 = (0.0, 0.2)$. Then the map ϕ_T from $\hat{T}$ to T is given by

$$\phi_T(\hat{x}, \hat{y}) = \begin{pmatrix} -0.1 & -0.2 \\ 0.1 & -0.1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix} = D \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + c.$$

Note that $\phi_T^{-1}(x,y) = D^{-1}((\hat{x},\hat{y}) - c)$ and

$$D^{-1} = \frac{1}{3}\begin{pmatrix} -10 & 20 \\ -10 & -10 \end{pmatrix}.$$

The basis function on T associated with z_1 is formally $\hat{\psi}_1 \circ \phi_T^{-1}$. So for $(x,y) \in T$ we have from ϕ_T^{-1}, the corresponding point in $\hat{T}$ is $\hat{x} = \frac{1}{3}(-10x + 20y - 4)$ and $\hat{y} = \frac{1}{3}(-10x - 10y + 5)$. Thus, since $\hat{\psi}_1 = 1 - \hat{x} - \hat{y}$,

$$\psi_1 = 1 - \frac{1}{3}(-10x + 20y - 4) - \frac{1}{3}(-10x - 10y + 5) = \frac{2}{3} + \frac{20}{3}x - \frac{10}{3}y.$$

From the formal expression for ψ_1, we have $\nabla\psi_1 = D^{-T}\nabla\hat{\psi}_1$. $\nabla\hat{\psi}_1 = (-1,-1)^T$ so

$$\nabla\psi_1 = \frac{1}{3}\begin{pmatrix} -10 & -10 \\ 20 & -10 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 20 \\ -10 \end{pmatrix}.$$

By definition $\psi(x,y) = a + bx + cy$ and it satisfies $\psi(z_1) = 1$ and $\psi(z_2) = \psi(z_3) = 0$. Using this we get the three equations:

$$\begin{aligned} a + b(0.2) + c(0.3) &= 1 \\ a + b(0.1) + c(0.4) &= 0 \\ a + b(0.0) + c(0.2) &= 0 \end{aligned}$$

and solving, we get $a = \frac{2}{3}$, $b = \frac{20}{3}$ and $c = -\frac{10}{3}$. From this we can compute directly that

$$\nabla\psi_1 = \begin{pmatrix} \frac{20}{3} \\ -\frac{10}{3} \end{pmatrix}.$$

Both methods for constructing ψ_1 and $\nabla\psi_1$ produce the same results. (Yea!)