

## Calculus Basics for M572

**Continuous Functions:** Let  $\Omega \subseteq \mathbb{R}^n$  be an open set, then  $C^k(\Omega)$  is the set of all functions defined on  $\Omega$  such that the function and its first  $k$  derivatives are continuous on  $\Omega$ . If  $k = 0$  then  $\Omega$  can be a closed (or partially closed) set. If  $\Omega$  is partially open then without further assumptions we cannot assume that if  $f \in C^k(\Omega)$  then  $f$  or any derivative of  $f$  up to  $k$  is bounded on  $\Omega$ . Example:  $f(x) = \sqrt{x}$  is in  $C([0, 1])$ ,  $C([0, \infty))$  and  $C^\infty((0, \infty))$ .  $f'$  is not in  $C([0, 1])$  as  $f'$  does not exist at  $x = 0$ .

**Taylor's Theorem:** Let  $a \in \mathbb{R}$  and let  $I$  be an open interval containing  $a$ . Let  $f \in C^{n+1}(I)$ . Then for all  $x \in I$ ,  $f(x) = P_n(x) + R_n(x)$  where

$$\begin{aligned} P_n(x) &= f(a) + f'(a) \cdot (x - a) + \frac{1}{2}f''(a) \cdot (x - a)^2 + \dots + \frac{1}{n!}f^{(n)}(a) \cdot (x - a)^n \\ &= \sum_{k=0}^n \frac{1}{k!}f^{(k)}(a) \cdot (x - a)^k, \end{aligned}$$

and

$$R_n(x) = \int_a^x \frac{1}{n!}(x - t)^n f^{(n+1)}(t) dt = \frac{(x - a)^{n+1}}{(n + 1)!}f^{(n+1)}(z),$$

where  $z$  is between  $a$  and  $x$ . Note that  $R_n(a) = 0$  by definition.  $P_n$  is called the Taylor Polynomial of degree  $n$  for  $f$  centered at  $x = a$  and  $R_n$  is called the Remainder.

Taylor's Theorem is used in both the development and analysis of numerical methods. When we replace  $f$  in an expression by  $P_n + R_n$ , the resulting expression involving  $P_n$  often leads to a method, while the part involving  $R_n$  leads to a term for the error in the method. In analyzing the error from the Remainder term it is important to be able to bound the error. Typically this involves finding a value  $M \geq 0$  such that  $|f^{(n+1)}(z)| \leq M$  for all  $z \in I$ . Such a  $M$  can be found by careful analysis involving derivatives etc. or, since it is just a constant, can be found by breaking an expression into parts and bounding each part.

**Mean Value Theorem:** (This can be thought of as a special case of Taylor's Theorem) Let  $a < b$ ,  $f \in C([a, b])$  and  $f' \in C((a, b))$ . Then there is a  $c \in (a, b)$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

**Mean Value Theorem for Integrals:** Let  $a < b$  and  $g, h \in C([a, b])$ . Suppose that  $h$  does not change sign in the interval  $[a, b]$  then there exists a  $c \in (a, b)$  such that

$$\int_a^b g(t)h(t) dt = g(c) \int_a^b h(t) dt.$$

If  $h$  does change sign then often an application of Taylor's Theorem can help things out. For example, let  $a = 0$ ,  $b = 1$  and  $h(t) = t - \frac{1}{2}$ . Then if  $g \in C^1(a, b)$  then we can write

$g(t) = g(\frac{1}{2}) + g'(z)(t - \frac{1}{2})$  for some  $z$  (depending on  $t$ ) and so

$$\begin{aligned} \int_0^1 g(t)h(t) dt &= \int_0^1 g(\frac{1}{2})h(t) dt + \int_0^1 g'(z(t))(t - \frac{1}{2})^2 dt \\ &= 0 + g'(z(c)) \int_0^1 (t - \frac{1}{2})^2 dt, \end{aligned}$$

where  $0 < c < 1$  since  $(t - \frac{1}{2})^2$  does not change sign in  $[0, 1]$ .

**Taylor's Theorem in  $m$  variables:** Many results for 1 variable don't extend into two or more variables. However, Taylor's Theorem does. Let  $a \in \mathbb{R}^m$  and let  $\Omega \subseteq \mathbb{R}^m$  be an open set containing  $a$ . Let  $f \in C^{n+1}(\Omega)$ . Then for every  $x \in \Omega$  such that the line segment from  $a$  to  $x$  is in  $\Omega$ , there is a point  $z$  on that segment such that

$$\begin{aligned} f(x) &= P_n(x) + R_n(x) \\ f(x) &= \sum_{k=0}^n \frac{1}{k!} D^k f(a) (x - a)^k + \frac{1}{(n+1)!} D^{n+1} f(z) (x - a)^{n+1}, \end{aligned}$$

where

$$D^k f(a)(x - a)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^k f(a)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_m^{\alpha_m}} (x_1 - a_1)^{\alpha_1} \dots (x_m - a_m)^{\alpha_m}.$$

Note  $\alpha$  is a multiindex where  $\alpha \in \mathbf{Z}^m$ ,  $\alpha_i \geq 0$ ,  $|\alpha| = \sum_{i=1}^m \alpha_i$  and  $\alpha! = \alpha_1! \alpha_2! \dots \alpha_m!$ . In the case of just 2 variables  $(x, y)$ , we have

$$D^k f(a)(x - a)^k = \sum_{i=0}^k \frac{k!}{i!(k-i)!} \frac{\partial^k f(a)}{\partial x^i \partial y^{k-i}} (x - a_1)^i (y - a_2)^{k-i}.$$

And we get, for  $n = 2$ ,

$$\begin{aligned} P_n(x, y) &= f(a) + \frac{\partial f(a)}{\partial x} (x - a_1) + \frac{\partial f(a)}{\partial y} (y - a_2) + \frac{1}{2} \frac{\partial^2 f(a)}{\partial x^2} (x - a_1)^2 \\ &+ \frac{\partial^2 f(a)}{\partial x \partial y} (x - a_1)(y - a_2) + \frac{1}{2} \frac{\partial^2 f(a)}{\partial y^2} (y - a_2)^2. \end{aligned}$$