MIDTERM EXAM

Carefully justify all answers. You will not receive any credit for correct answers without reasons or with wrong reasons.

I. Do 4 of the following 5 problems. (15 points each)

1. Let $A \in \mathcal{A}_n$ with $\det A \neq 0$. Show that $A^T A$ is symmetric and positive definite.

2. Show that for every norm on $\mathcal{A}_n$, every $A \in \mathcal{A}_n$, and every $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $\text{cond}(\alpha A) = \text{cond}(A)$.

3. For $\mathbb{R}^2$ is $||x|| \stackrel{\text{def}}{=} |x_1|$ a vector norm? Why or why not?

4. For the following matrices

$$A_1 = \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 3 & 0 \\ 6 & 3 \end{pmatrix}$$

let J_i for $i = 1, 2$ be the iteration matrix for Jacobi's method. Find J_i and compute $||J_i||_\infty$ and $\rho(J_i)$. For which matrices does Jacobi iteration converge?

5. If A is not invertible but has an LU-factorization, is L or U invertible? Explain.

II. Do 2 of the following 4 problems. (20 points each)

1. Let $A \in \mathcal{A}_n$ be an orthogonal matrix (i.e. $A^T = A^{-1}$). Show that $\text{cond}_2(A) = 1$.

2. Show that the matrix norm

$$||A||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

is a subordinate matrix norm induced by the vector norm

$$||x||_\infty = \max_{1 \leq i \leq n} |x_i|.$$

3. Compare the cost of solving $A^2 u = f$ using two different methods. (1) Compute A^2 and then use LU-factorization to solve $(A^2)u = f$. (2) Compute the LU-factorization of A and solve the two problems $Aw = f$ and $Au = w$. Would the results be different if Cholesky factorization could be used? What if A has bandwidth p?

4. Let a, b and c be real numbers. If $a > 0$ and $ac - b^2 > 0$ show that the matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite and compute its Cholesky factorization.

Equations:

$$\text{cond}(A) = ||A||\,||A^{-1}||$$

$$||x||_2 = \sqrt{x^T x}$$

$$\text{Jacobi } Du^{k+1} = (E+F)u^k + b$$

$$\text{subordinate matrix norm } ||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||} = \max_{||x||=1} ||Ax||$$

Operation Counts: (2nd is for bandwidth p)

LU-factorization = $2n^3/3$, $p^2 n$

Cholesky factorization = $n^3/3$, $p^2 n/2$

Back/Forward Substitution = n^2, np