You can use any written resources, but please work alone.

32. For each of the following sequences, determine the order of convergence for the sequence.
   (a) \( x^k = 1/k, \ x^* = 0 \).
   (b) \( x^k = 1 + 1/k!, \ x^* = 1 \).
   (c) \( x^k = 2^{-2^k}, \ x^* = 0 \).
   (d) \( x^k = 10^{-k^3}, \ x^* = 0 \).

33. **Bisection Method:** Given a continuous function \( f \), and numbers \( a \) and \( b \) with \( a < b \) and \( f(a)f(b) < 0 \). Consider the sequence of values \( x(k), a(k) \) and \( b(k) \) produced by the following algorithm:

   a(0) = a
   b(0) = b
   for k = 0, 1, 2, ...
   \( x(k) = (a(k)+b(k))/2 \)
   if \( (f(x(k))*f(a(k))<=0) \) then
   \( a(k+1) = a(k) \)
   \( b(k+1) = x(k) \)
   else
   \( a(k+1) = x(k) \)
   \( b(k+1) = b(k) \)
   endif
   end

   Assume that \( f(x(k)) \neq 0 \) for all \( k \).

   (a) Show that \( f \) has a root \( x^* \) in the interval \((a, b)\).
   (b) Show that \( f \) has a root \( x^* \) in the interval \((a(k), b(k))\) for each \( k \). (Not necessarily always the same root for each \( k \)).
   (c) Show that \( b(k+1) - a(k+1) = \frac{1}{2}(b(k) - a(k)) \).
   (d) Show that the sequence \( \{x(k)\} \) converges to a root \( x^* \) of \( f \).
   (e) Determine the order of convergence for Bisection.

34. (§5.7 #3) For each of the functions \( f_1(x) = x, f_2(x) = x^2 + x \) and \( f_3(x) = e^x - 1 \) answer the following:

   (a) What is \( f'(x) \) at the root \( x^* = 0 \)?
   (b) Let \( D = (-a, a) \). For what values of \( a \) and \( \rho > 0 \) is \( |f'(x)| \geq \rho \) on \( D \).
   (c) What is a Lipschitz constant \( \gamma \) for \( f'(x) \) on \( D \); i.e. what is a bound on \( |(f'(x) - f'(0))/x| \) in this interval?
   (d) What region of convergence of Newton’s method applied to \( f(x) \) is predicted by Theorem 2.4.3 (Newton Convergence Theorem); i.e. what value of \( \eta \) does the theorem produce?
   (e) What is the largest interval \((b, c), b < 0 < c \) such that Newton’s method applied to \( f(x) \) starting with any \( x^0 \in (b, c) \) actually converges to \( x^* = 0 \)? (You don’t have to prove it)
35. (§5.7 #6) Prove that the conditions and conclusions of the Kantorovich theorem (5.3.1) are satisfied for the quadratic:

\[ f(t) = \frac{\gamma}{2} t^2 - \frac{t}{\beta} + \frac{\eta}{\beta}, \]

with \( t_0 = 0 \).

36. The system of equations:

\[
\begin{align*}
2(x_1 + x_2)^2 + (x_1 - x_2)^2 - 8 &= 0 \\
5x_1^2 + (x_2 - 3)^2 - 9 &= 0,
\end{align*}
\]

has a solution \( x^* = (1, 1) \).

(a) Carry out two iterations of Newton’s method starting with \( x^0 = (2, 0) \).

(b) Carry out two iterations of Broyden’s method starting with \( x^0 = (2, 0) \) and \( A^0 = I_2 \) (2x2 identity).

37. Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) satisfy all the conditions of the Newton Convergence Theorem (5.2.1). Under what condition(s) on \( x^0 \) does the following iteration converge to \( x^* \):

\[ x^{k+1} = x^k - J(\cdot)^{-1}F(x^k)? \]