Math 571 – Fall 2007 – Homework #11
Due Tuesday, December 4

You can use any written resources, but please work alone.

32. For each of the following sequences, determine the order of convergence for the sequence.

    (a) $x^k = 1/k$, $x^* = 0$.

    (b) $x^k = 1 + 1/k!$, $x^* = 1$.

    (c) $x^k = 2^{-2^k}$, $x^* = 0$.

    (d) $x^k = 10^{-k^3}$, $x^* = 0$.

33. **Bisection Method:** Given a continuous function f, and numbers a and b with $a < b$ and $f(a)f(b) < 0$. Consider the sequence of values $x(k)$, $a(k)$ and $b(k)$ produced by the following algorithm:

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a(0) = a
b(0) = b
for k = 0, 1, 2, ...
   x(k) = (a(k)+b(k))/2
   if (f(x(k))*f(a(k))<=0) then
      a(k+1) = a(k)
      b(k+1) = x(k)
   else
      a(k+1) = x(k)
      b(k+1) = b(k)
   endif
end
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    Assume that $f(x(k)) \neq 0$ for all k.

    (a) Show that f has a root x^* in the interval (a, b).

    (b) Show that f has a root x^* in the interval $(a(k), b(k))$ for each k. (Not necessarily always the same root for each k).

    (c) Show that $b(k+1) - a(k+1) = \frac{1}{2}(b(k) - a(k))$.

    (d) Show that the sequence $\{x(k)\}$ converges to a root x^* of f.

    (e) Determine the order of convergence for Bisection.

34. (§5.7 #3) For each of the functions $f_1(x) = x$, $f_2(x) = x^2 + x$ and $f_3(x) = e^x - 1$ answer the following:

    (a) What is $f'(x)$ at the root $x^* = 0$?

    (b) Let $D = (-a, a)$. For what values of a and $\rho > 0$ is $|f'(x)| \geq \rho$ on D.

    (c) What is a Lipschitz constant γ for $f'(x)$ on D; i.e. what is a bound on $|(f'(x) - f'(0))/x|$ in this interval?

    (d) What region of convergence of Newton's method applied to $f(x)$ is predicted by Theorem 2.4.3 (Newton Convergence Theorem); i.e. what value of η does the theorem produce?

    (e) What is the largest interval (b, c), $b < 0 < c$ such that Newton's method applied to $f(x)$ starting with any $x^0 \in (b, c)$ actually converges to $x^* = 0$? (You don't have to prove it)

35. (§5.7 #6) Prove that the conditions and conclusions of the Kantorovich theorem (5.3.1) are satisfied for the quadratic:

$$f(t) = \frac{\gamma}{2}t^2 - \frac{t}{\beta} + \frac{\eta}{\beta},$$

with $t_0 = 0$.

36. The system of equations:

$$\begin{aligned} 2(x_1 + x_2)^2 + (x_1 - x_2)^2 - 8 &= 0 \\ 5x_1^2 + (x_2 - 3)^2 - 9 &= 0, \end{aligned}$$

has a solution $x^* = (1, 1)$.

(a) Carry out two iterations of Newton's method starting with $x^0 = (2, 0)$.

(b) Carry out two iterations of Broyden's method starting with $x^0 = (2, 0)$ and $A^0 = I_2$ (2x2 identity).

37. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfy all the conditions of the Newton Convergence Theorem (5.2.1). Under what condition(s) on x^0 does the following iteration converge to x^*:

$$x^{k+1} = x^k - J(x^0)^{-1}F(x^k)?$$