

HOMEWORK #7

1. For the model $\frac{dx}{dt} = f(x)$ where $f(x) = ax(x - b)(c - x)$:

- (a) If we assume x has units P (for population) and time has units T , then we see that both b and c must have units P and thus for a we have $P/T = [a]P^3$ so $[a] = 1/(P^2T)$.

General Info: Consider the growth rate $\frac{1}{x} \frac{dx}{dt} = a(x - b)(c - x)$. The growth is positive when $b < x < c$ and negative for $x < b$ or $x > c$. In any population model b and c are some type of extreme population limits and might be close in value or several magnitudes apart depending on the situation. For a , we can look at the growth rate for 'typical' x value(s), like $x = (b + c)/2$ or, more generally, $x = \lambda b + (1 - \lambda)c$. For these typical values we might have a desired growth rate and thus can determine a value for a .

Example 1: $x(t)$ is the number of animals in a herd. If the herd is too small ($x < b$) they are more vulnerable to predators and die off or if the herd is too large ($x > c$) the area can't support the entire herd and so it begins to die. Thus b is the minimum herd size (around 10-20), c is the land capacity (around 100-200). If we assume a typical herd is 50 and has a growth rate of 10%, then we have $0.1 = a(50 - b)(c - 50)$ or, with the given ranges of b and c , we get a in the range of 1.7×10^{-5} .

Example 2: $x(t)$ is the number of animals in a pack that hunts together. If the pack is too small ($x < b$) then they are not effective and can't get enough food to support themselves. If the pack is too large ($x > c$) then either the pack is so big it scares away its prey or whatever they hunt isn't large enough or plentiful enough to feed everyone. In this case we might have b around 3-5 and c around 15-20. If we have a low growth rate of 5% in a pack of 8-10, then a is around 0.001.

Example 3: $x(t)$ is the size (mass, volume, or diameter) of a single-cell creature in an environment containing nutrients and hazardous compounds. If it is too small, then the surface area to mass ratio is large so the damage from the environment is more than can be repaired. If it is too large, the surface area to mass ratio is too small for it to absorb the nutrients it needs to survive. We might have b around 1mm, c around 20mm and for a typical size of 8mm we might see 20% growth, thus a is around 0.002.

- (b) If $0 < x(0) < b$, $f(x) < 0$ so $x' < 0$ and the population decreases towards 0, if $b < x(0) < c$, $f(x) > 0$ so $x' > 0$ and the population increases towards c , and if $c < x(0)$, $f(x) < 0$ so $x' < 0$ and the population decreases towards c .
- (c) There are 3 fixed points 0, b and c . 0 and c are attracting and b is repelling.
- (d) As we increase the harvesting rate, the type of the fixed point doesn't change, but the one that was at 0 increases, the one that was at b decreases and the one that was at c increases. The maximum practical value for the harvesting is when the 0 and b points come together as one fixed point (repelling) and then disappear for higher harvesting levels leaving only the repelling point that was c .

2. Starting with the predator-prey system:

$$x' = x(a - by), \quad y' = -y(c - dx).$$

- (a) After substituting $x = AX + c/d$, $y = BY + a/b$ and $t = CT$, we get $x' = A/CX'$, $y' = B/CY'$ where X' and Y' are with respect to T . Plugging these into our equations and simplifying, we get

$$X' = -\frac{bBC}{A}(AX + c/d)Y, \quad Y' = \frac{dAC}{B}(BY + a/b)X,$$

or

$$X' = -bBC(X + \frac{c}{dA})Y, \quad Y' = dAC(Y + \frac{a}{bB})X.$$

- (b) To make the system as simple as possible, we want to make as many of the coefficients equal to 1 as possible. The first obvious choices are to take $A = c/d$ and $B = a/b$, then the system reduces to

$$X' = -aC(X + 1)Y, \quad Y' = cC(Y + 1)X.$$

Then we can either choose $C = 1/a$ or $C = 1/c$. Take $C = 1/a$ and let $\lambda = c/a$, then we have

$$X' = -(X + 1)Y, \quad Y' = \lambda(Y + 1)X.$$

If the units of x and y are denoted by Q , and time is measured in units T , then a and c have units $1/T$ and b and d have units $1/(QT)$, so with our change of variables A and B have units Q and C has units T . So X , Y , T and λ would be unitless.

- (c) Assuming X and Y are small (so that XY is ignorable), we can now simplify the system to be:

$$X' = -Y, \quad Y' = \lambda X.$$

Now, dividing the expression for Y' by the expression for X' , we get

$$\frac{dY}{dX} = \frac{Y'}{X'} = -\lambda \frac{X}{Y}.$$

or

$$YdY = -\lambda XdX.$$

Integrating, we get $\frac{1}{2}Y^2 = -\frac{\lambda}{2}X^2 + C$. Using the initial conditions $X(0) = X_0$ and $Y(0) = Y_0$, we get that $C = \frac{1}{2}(Y_0^2 + \lambda X_0^2)$. Thus we get

$$Y^2 + \lambda X^2 = (Y_0^2 + \lambda X_0^2).$$

This is the equation of an ellipse centered at the origin, with axes of length $L = \sqrt{Y_0^2 + \lambda X_0^2}$ and L/λ .