HOMEWORK #9

Due Monday, November 30

Each problem is worth 5 points and will be added to your HW total.

1. With an appropriate change of variable, we can transform the traffic flow problem into the following:
$$\rho_t + (1-2\rho)\rho_x = 0, \qquad \rho(x,0) = f(x),$$
where now ρ is the 'normalized' density with $0 \le \rho \le 1$. The solution of this problem is
$$\rho(x,t) = f(x - t(1 - 2\rho(x,t))),$$
but it only can be solved for ρ explicitly under certain circumstances. Use this form of the traffic flow problem to answer the following questions.

   (a) Find a closed form expression for $\rho(x,t)$, for $t > 0$ when
$$f(x) = \begin{cases} 1, & x \le 0 \\ (1-x), & 0 \le x \le 1 \\ 0, & x \ge 1 \end{cases}.$$
Hint: follow the transition points which are initially at $x = 0$ and $x = 1$ and then figure out what happens to the solution in the regions between them and outside them.

   (b) Find a closed form expression for $\rho(x,t)$, for $t > 0$ and before the shock forms, when
$$f(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}.$$

2. (Fourier's Law for heat flow) Let J be the rate at which heat is flowing through a cross-section of area A in the positive x-direction. Fourier's Law says that J is proportional to the change in temperature, or if $u(x)$ represents the temperature at position x, then
$$J = -kA\frac{du}{dx},$$
where k is heat conductivity constant and A is the cross-section area. Consider the *steady-state* case, i.e. where $J' = 0$, the equation then becomes
$$\frac{du}{dx} = -\frac{J}{kA},$$
where J is now a constant.

   (a) Assuming J, k and A are constant, solve for the temperature $u(x)$ with $u(0) = u_0$. Draw a sketch of $u(x)$. Suppose $L > 0$, what is $u(L)$? Solve for J in terms of the other values (k, A, u_0, $u(L)$).

   (b) In a furnace the inner wall is at 500 °C and the outer wall is at 100 °C. If the wall has area 3 m^2, is 1 m thick and is made of asbestons ($k = 0.113$), find the amount of heat escaping from the furnace through the wall. Hint: Assume steady state and then solve for J.

(c) In a house the inner wall is at 22 °C and the outer wall is at -18 °C. The wall has area 18 m^2 and is made of an outer layer of brick ($k = 0.45$) with thickness 0.2 m and an inner layer of wood ($k = 0.15$) with thickness 0.3 m. Find the amount of heat escaping from the house through the wall. Hint: J is constant throughout, but k now changes. Introduce a new variable for the temperature at the point between the brick part and the wood part.

3. For the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial t^2}$$

with $0 < x < L$, $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$.

Use separation of variables to find the general form of the solution. (Substitute $u(x,t) = X(x) \cdot T(t)$ and solve the resulting ordinary differential equations)