Homework #3
Due Friday, September 11

1. Consider the following model for the spread of a rumor:

\[ R_{n+1} = R_n + \frac{a R_n}{M + R_n} (N - R_n) - b R_n, \]

where \( N \) is the total population, \( a \) is a spreading rate, \( M \) is a radius of influence and \( b \) is a forgetting rate. (All constants are \( \geq 0 \)).

(a) Determine the fixed points for this model with \( b = 0 \) and with \( b > 0 \).
(b) Determine conditions on \( a \) and \( M \) in the case when \( b = 0 \) so that \( N \) is an attracting fixed point.
(c) Determine conditions on \( a \), \( M \) and \( b \) so that \( f N \) is an attracting fixed point for this system \( 0 < f \leq 1 \). (Your answers will depend on \( f \))

2. Construct an SIS model for the common cold. Assume the mechanics behind the susceptibles becoming infected is simple mixing and that people recover (and become susceptible) after some average recovery time. Include natural births and deaths in either or both state as you think is appropriate.

Assuming there is no births or deaths in the system, show that \( S + I \) remains constant in your model.

Also without births and deaths, determine the fixed point(s) for the system.

With births and deaths included, how do the fixed point(s) change?

3. To review the finding of eigenvalues and eigenvectors and their properties. Let

\[ A = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}. \]

The eigenvalues and eigenvectors come in pairs \((\lambda, v)\) where \( \lambda \) is a (complex) scalar and \( v \) is a vector, with \( Av = \lambda v \). For a \( n \times n \) matrix (\( 2 \times 2 \) for \( A \)) we always have \( n \) eigenvalues (counting multiplicities), but may have fewer than \( n \) eigenvectors. Also, if \( v \) is an eigenvector then so is any multiple of \( v \).

(a) Eigenvalues first: Since \( Av = \lambda v \), we have \( (A - \lambda I)v = 0 \) (\( I \) is the \( 2 \times 2 \) identity matrix). For this to have a solution \( v \neq 0 \), we must have \( \det(A - \lambda I) = 0 \). This last equation is the characteristic equation for the eigenvalues of \( A \).

Recall for \( 2 \times 2 \) matrices,

\[ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc. \]

Compute

\[ \det(A - \lambda I) = \det \begin{pmatrix} 0.8 - \lambda & 0.1 \\ 0.2 & 0.9 - \lambda \end{pmatrix} \]

and then solve the corresponding quadratic equation for the two eigenvalues \( \lambda_1 \) and \( \lambda_2 \). (You should get the values 1 and 0.7)
(b) Eigenvalues next: For each eigenvalue, you go through the same process to find the corresponding eigenvector. First, let \( v = (v_1, v_2)^T \) and write out the system \( (A - \lambda I)v = 0 \) using the particular eigenvalue.

This system should have multiple solutions, so we just need to pick one of them. In the 2 \( \times \) 2 case, you can just use one of the equations to get the relationship between \( v_1 \) and \( v_2 \), then pick a value for \( v_2 \) (not 0) and solve for \( v_1 \). Then you’ve found an eigenvector.

Find the eigenvectors corresponding to the eigenvalues you found in part (a). (For \( \lambda = 1 \) you should get \( v = (0.5, 1) \) or some multiple, and for \( \lambda = 0.7 \) you should get \( v = (-1, 1) \) or some multiple).

(c) Eigenvectors as basis: Typically you’ll get \( n \) distinct eigenvectors for an \( n \times n \) matrix and those eigenvectors form a basis for all \( n \)-vectors. That means any vector can be written as a linear combination of the eigenvectors.

Let \( v \) and \( w \) be the two eigenvectors that you computed in part (b). Let \( u \) be any 2-vector, then we can find constants \( a \) and \( b \) such that \( av + bw = u \), by simply writing out the corresponding linear system and then solving it for \( a \) and \( b \).

Write out the vector \((1, 0)^T\) in terms of \( v \) and \( w \). Repeat for the vector \((0, 1)^T\). (Your exact results will depend on how \( v \) and \( w \) are scaled)

(d) Eigenvalues for powers: If \((\lambda, v)\) are an eigenpair for a matrix \( A \), i.e. \( Av = \lambda v \), then \( A^2v = A(Av) = A(\lambda v) = \lambda (Av) = \lambda^2 v \). In general, for any natural number \( m \) we have \( A^m v = \lambda^m v \). So, if \( Av = \lambda_1 v \) and \( Aw = \lambda_2 w \) then for scalars \( a \) and \( b \),

\[
A^m (av + bw) = a\lambda_1^m v + b\lambda_2^m w.
\]

Calculate \( A^3(1, 0)^T \) by two different means. First, calculate \( A^3 \) and then multiply by \((1, 0)^T\). Second, use the results of (c) and the above formula to calculate in terms of the eigenvalues and eigenvectors. (You should get the same answer by both method).