Math 142 – Quiz 7 Solutions

Quiz Info: Mean 22.33 (74%), Standard Deviation 3.63, 2 As, 7Bs, 9Cs

1. 1. $R = 1$, 2. $R = 1$, 3. $R = 1$, 4. $R = \infty$, 5. $R = \infty$, 6. $R = \infty$

2. (a) Using the ratio test:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| (-1)^{n+1} \frac{x^{n+1}}{2^{n+1}} / (-1)^n \frac{x^n}{2^n} \right| \\ &= \lim_{n\to\infty} \frac{12}{|} x| = \frac{1}{2}|x|. \end{aligned}$$

Series converges when $L < 1$ or $\frac{1}{2}|x| < 1$ or $|x| < 2$. Thus $R = 2$.

$$\text{Bonus: } \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{-x}{2} \right)^n = \frac{1}{1 - \frac{-x}{2}} = \frac{2}{2+x}.$$

(b) Using the ratio test:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{3^{n+2}}{(n+1)!} (x-1)^{n+1} / \frac{3^{n+1}}{n!} (x-1)^n \right| \\ &= \lim_{n\to\infty} 3|x-1| \frac{1}{n+1} = 0. \end{aligned}$$

Series converges when $L < 1$, so it converges for all x. Thus $R = \infty$.

$$\text{Bonus: } \sum_{n=0}^{\infty} \frac{3^{n+1}}{n!} (x-1)^n = 3 \sum_{n=0}^{\infty} \frac{(3(x-1))^n}{n!} = 3e^{3(x-1)}.$$

(c) Using the ratio test:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| (-1)^{n+1} \frac{(n+1)8^{n+1}}{\sqrt{n+2}} (x+2)^{2n+2} / (-1)^n \frac{n8^n}{\sqrt{n+1}} (x+2)^{2n} \right| \\ &= \lim_{n\to\infty} \frac{n+1}{n} 8 \frac{\sqrt{n+1}}{\sqrt{n+2}} |x+2|^2 \\ &= 8|x+2|^2. \end{aligned}$$

Series converges when $L < 1$ or $8|x+2|^2 < 1$ or $|x+2| < \sqrt{1/8}$. Thus $R = \sqrt{1/8}$.

Bonus: No known simple form for this function

3. (a) We have $f(x) = \frac{1}{3+2x}$, $f'(x) = -\frac{2}{(3+2x)^2}$ and $f''(x) = \frac{8}{(3+2x)^3}$, so $f(0) = 1/3$, $f'(0) = -2/9$ and $f''(0) = 8/27$. Thus the Taylor series is

$$1/3 - 2/9x + 4/27x^2 + \ldots$$

Since $f(x) = \frac{1}{3} \frac{1}{1-(-2/3x)}$, we have the general form when the series converges of

$$f(x) = \frac{1}{3} \sum_{n=0}^{\infty} (-2/3x)^n = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^{n+1}} x^n,$$

and this can also be used to generate the first 3 terms.

(b) We have $f(x) = x^{1/2}$, $f'(x) = 1/2x^{-1/2}$, and $f''(x) = -1/4x^{-3/2}$, so $f(9) = 3$, $f'(9) = 1/6$ and $f''(9) = -1/108$. Thus the Taylor series is

$$3 + 1/6(x-9) - 1/216(x-9)^2 + \dots$$

The general expression for the nth term of this series is more complicated and for $n \geq 1$, it is

$$\frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \dots (2n-3)}{2^n 3^{2n-1} n!}(x-9)^n.$$

(c) We have $f(x) = e^x \sin x$, $f'(x) = e^x(\sin x + \cos x)$, $f''(x) = 2e^x \cos x$ and $f^{(3)}(x) = 2e^x(\cos x - \sin x)$, so $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$ and $f^{(3)}(0) = 2$. Thus the Taylor series is

$$x + x^2 + \frac{1}{3}x^3 + \dots$$

You can also get this by (carefully) multiplying $1 + x + \frac{1}{2}x^2$ by $x - \frac{1}{6}x^3$.

I wouldn't expect anyone to get this, but there is a way to get the general form of the series. First note that $e^{ix} = \cos x + i \sin x$ so that $\sin x$ is the imaginary part of e^{ix}. Thus $e^x \sin x$ is the imaginary part of

$$e^x e^{ix} = e^{(1+i)x} = \sum_{n=0}^{\infty} \frac{1}{n!}(1+i)^n x^n.$$

If use the binomial formula from 8.8 (which we didn't cover) we can expand $(1+i)^n$ and get all the imaginary terms.

4. By the remainder formula we have

$$|e^x - T_3(x)| \leq \frac{M_4}{4!}|x|^4,$$

where $|f^{(4)}(x)| \leq M_4$ for $0 \leq x \leq 0.5$. With $f(x) = e^x$, $f^{(4)}(x) = e^x$, and $|e^x| \leq e^{0.5}$ for $0 \leq x \leq 0.5$. So for $0 \leq x \leq 0.5$,

$$|e^x - T_3(x)| \leq \frac{e^{0.5}}{24}|x|^4$$

and in the worse possible case, we have (when $x = 0.5$)

$$|e^x - T_3(x)| \leq \frac{e^{0.5}}{24}(0.5)^4 = 0.0042935.$$