Math 142 – Quiz 6 – Solutions

1. (a) By direct calculation,

$$\lim_{n\to\infty}\frac{1+3n}{2-5n}=\lim_{n\to\infty}\frac{\frac{1}{n}+3}{\frac{2}{n}-5}=\frac{0+3}{0-5}=-\frac{3}{5}.$$

So it is convergent with limit $-\frac{3}{5}$.

(b) By using $f(x)$, where $a_n=f(n)$,

$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0.$$

(Obtained using L'Hôpital's Rule twice). Thus the sequence is convergent with limit 0.

(c) By comparison,

$$\frac{n-1}{n+1}\leq\frac{n+\cos(n)}{n+1}\leq 1 \text{ and } \lim_{n\to\infty}\frac{n-1}{n+1}=1$$

so by the Squeeze Theorem, the sequence $\left\{\frac{n+\cos(n)}{n+1}\right\}$ converges to 1.

2. (a) By the Ratio Test (or as a geometric series),

$$L=\lim_{n\to\infty}\left|(16\frac{3^{2n+2}}{7^{n+1}})/(16\frac{3^{2n}}{7^n})\right|=\lim_{n\to\infty}\left|\frac{3^2 3^{2n}}{3^{2n}}\right|\left|\frac{7^n}{7\,7^n}\right|=\frac{9}{7}.$$

The ratio is bigger than 1, so the series is divergent. Can also show using the Divergence Test since $\lim_{n\to\infty}a_n=\infty$.

(b) By the integral test

$$\int_2^\infty\frac{1}{x\ln x}\,dx=\lim_{t\to\infty}\int_2^t\frac{1}{x\ln x}\,dx=\lim_{t\to\infty}\ln(\ln x)|_2^t=\lim_{t\to\infty}\ln(\ln t)-\ln(\ln 2)=\infty.$$

So the integral diverges and thus by the Integral Test, the series also diverges.

(c) By comparison,

$$\lim_{n\to\infty}\frac{(n^3-3n+1)/(n^5+2n^3)}{1/n^2}=\lim_{n\to\infty}\frac{n^5-3n^3+n^2}{n^5+2n^3}=\lim_{n\to\infty}\frac{1-3n^{-2}+n^{-3}}{1+2n^{-2}}=1.$$

Since $\sum 1/n^2$ converges (p-series, $p=2>1$), by the Limit Comparison Test, the series converges.

(d) By the ratio

$$L=\lim_{n\to\infty}\left|\frac{(-10)^{n+1}/(n+1)!}{(-10)^n/n!}\right|=\lim_{n\to\infty}\frac{10}{n+1}=0.$$

Since the ratio is less than 1, by the Ratio Test, the series converges.

(e) With the limit

$$\lim_{n\to\infty}\frac{n}{\sqrt{n+1}}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{1+1/n}}=\infty.$$

The limit is not 0, so, by the Divergence Test, this series diverges. Can also compare to $\sum\sqrt{n}$ via Limit Comparison Test.

(f) We have

$$0 \leq \frac{\sin^2 n}{n\sqrt{n}} \leq \frac{1}{n^{3/2}}$$

The series $\sum \frac{1}{n^{3/2}}$ converges (p-series, $p = 3/2 > 1$) so by the (Direct) Comparison Test, the series converges.

(g) By the limit

$$\lim_{n\to\infty} \frac{1/n - 1/n^2}{1/n} = \lim_{n\to\infty} 1 - \frac{1}{n} = 1.$$

Since $\sum 1/n$ diverges (p-series, $p = 1 \leq 1$), by the Limit Comparison Test the series diverges. Can also use the integral test with $\int \frac{1}{x} - \frac{1}{x^2}\, dx = \ln x + \frac{1}{x}$. Can also use direct comparison with $\frac{1}{n} - \frac{1}{n^2} > \frac{1}{n-1}$.

3. By the ratio

$$L = \lim_{k\to\infty} \left| \frac{x^{2k+2}/((k+1)2^{k+1})}{x^{2k}/(k2^k)} \right| = \lim_{k\to\infty} |x^2| \left| \frac{k}{k+1} \right| |1/2| = |x^2|/2.$$

By the Ratio Test we have convergence when $L < 1$ or $|x^2|/2 < 1$ or $|x| < \sqrt{2}$. We have divergence when $|x| > \sqrt{2}$. When $|x| = \sqrt{2}$, $x^2 = 2$ and the terms in the series reduce to $\frac{1}{k}$ and the resulting series is divergent. So the original series is convergent only for $|x| < \sqrt{2}$ and is divergent otherwise.

4. (a) The sequence must converge to 0, otherwise by the Divergence test, the series would be divergent.

(b) Since for small x, $\sin(x) \leq x$, $0 \leq \sin(a_n) \leq a_n$ (or $\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = \lim_{x\to 0} \frac{\sin x}{x} = 1$). Thus by the Comparison (or Limit Comparison) Test, $\sum \sin(a_n)$ is convergent (to something less than 1).