Math 142 – Quiz 5 – Solutions

1. **Bungee jumping.** Divide the distance stretched (100m to 150m) into pieces of size $\Delta x = 50/n$. Then for one piece we stretch the cord from x_i to $x_i + \Delta x$. Assuming the force is a constant value of $0.1x_i + 0.004x_i^3$ at that time, the work for the stretch in that piece is $(0.1x_i + 0.004x_i^3)\Delta x$ and the total work is approximately $\sum_{i=1}^{n}(0.1x_i + 0.004x_i^3)\Delta x$. Taking the limit to get the actual value
$$\text{Work} = \int_0^{50} (0.1x + 0.004x^3)\, dx = \frac{0.1}{2}x^2 + 0.001x^4|_0^{50} = 6375 \text{ J or N-m.}$$

2. **Anchor raising.** Divide the distance lifted (80 ft) into pieces of size $\Delta x = 80/n$. Then for one piece we lift the anchor and remaining chain from x_i to $x_i + \Delta x$. The weight we are lifting is approximately $1000 + 3(80 - x_i)$ lbs, so the work required is $(1250 - 3x_i)\Delta x$ and the total work is approximately $\sum_{i=1}^{n}(1250 - 3x_i)\Delta x$. The limit (as $n \to \infty$) gives us the actual value
$$\text{Work} = \int_0^{80} (1250 - 3x)\, dx = 1250x - \frac{3}{2}x^2|_0^{80} = 89,600 \text{ ft-lbs.}$$

3. **Tank pumping.** Since we are only pumping the top half of the tank, slice the top half horizontally into slices with thickness $\Delta x = 2/n$. Take x_i to be the distance from the mid-plane of the tank (up), then the volume of the slice between x_i and $x_i + \Delta x$ is $V = 2\sqrt{4 - x_i^2} \cdot 20 \cdot \Delta x$. The weight of this volume is $M = g800V$ and the distance it needs to be pumped is $12 - x_i$. So the work needed to pump this slice is then $3200g(12 - x_i)\sqrt{4 - x_i^2}\Delta x$. Add these up from $i = 1$ to n to get the approximate work needed and take the limit to get
$$\text{Work} = 3200g \int_0^{2} (12 - x)\sqrt{4 - x^2}\, dx = 3200g(12\pi - \frac{8}{3}) = 1.0986 \times 10^7 \text{ J or N-m.}$$

4. **Gate Pressing.** Slice the gate horizontally into strips of height $\Delta x = 8/n$. Take x_i to be the distance from the top of the gate (down). Then the area of the slice between x_i and $x_i + \Delta x$ is approximately $x_i \Delta x$ and its depth is $32 + x_i$. Then the approximate force is $\delta(32 + x_i)x_i\Delta x$. The approximate total force is $\sum_{i=1}^{n} \delta(32 + x_i)x_i\Delta x$ and the limit gives us
$$\text{Hydrostatic Pressure} = \delta \int_0^{8} (32 + x)x\, dx = \delta(16x^2 + \frac{1}{3}x^3|_0^8 = 74,667 \text{ ft-lbs.}$$

5. **Boomerang Centering.** The curves intersect at $x = -2$ and $x = 2$. Slice the region vertically into rectangles of width $\Delta x = 4/n$. A rectangle at the point x_i has center of mass $(x_i, (4 - x_i^2 + 3\cos\frac{\pi}{4}x_i)/2)$ with mass (or area) $(4 - x_i^2 - 3\cos\frac{\pi}{4}x_i)\Delta x$. To get the y-coordinate of the centroid, we need the moment with respect to the x-axis which is approximately $\sum_{i=1}^{n}(4 - x_i^2 - 3\cos\frac{\pi}{4}x_i)(4 - x_i^2 + 3\cos\frac{\pi}{4}x_i)\Delta x$. Letting $n \to \infty$, we get
$$\text{Moment} = \int_{-2}^{2} (4 - x^2)^2 - 9\cos^2\frac{\pi}{4}x\, dx = 16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 - 9(\frac{1}{2}x + \frac{1}{\pi}\sin\frac{\pi}{2}x)|_0^2 = \frac{121}{15}.$$
The total mass is approximately $\sum_{i=1}^{n}(4 - x_i^2 - 3\cos\frac{\pi}{4}x_i)\Delta x$ which in the limit is
$$\text{Mass} = \int_{-2}^{2} 4 - x^2 - 3\cos\frac{\pi}{4}x\, dx = 4x - \frac{1}{3}x^3 - \frac{12}{\pi}\sin\frac{\pi}{4}x|_{-2}^2 = \frac{32}{3} - \frac{24}{\pi}.$$
The y-coordinate is the ratio of the moment to the mass, which is 2.6647.

6. **Money Losing.** Divide the 10 year period into pieces of size $\Delta t = 10/n$. Over a period from t_i to $t_i + \Delta t$ the approximate loss to the employee is $(40000e^{0.1t_i} - 30000)\Delta t$. So the total loss is about $\sum_{i=1}^{n}(40000e^{0.1t_i} - 30000)\Delta t$, which in the limit, gives

$$\text{Loss} = \int_0^{10} 40000e^{0.1t} - 30000\, dt = 10000(40e^{0.1t} - 3)|_0^{10} = \$687,312.73.$$

7. **Fair Gambling.** Divide the range of values into pieces of size $\Delta v = \pi/n$. The probability of getting a value in the range from v_i to $v_i + \Delta v$ is $\Delta v/\pi$ and the corresponding payoff is approximately $v_i \sin v_i$. Thus the average payoff is about $\sum_{i=1}^{n} v_i \sin v_i \frac{1}{\pi}\Delta v$. As $n \to \infty$, this gives us

$$\text{Average Payoff} = \frac{1}{\pi}\int_0^{\pi} v \sin v\, dv = \frac{1}{\pi}(\sin v - v\cos v)|_0^{\pi} = \$1.$$