Math 142 – Quiz 4 – Solutions

1. Determine if each of the following integrals is convergent or divergent. If it is convergent, evaluate the integral.

$$\text{(a)} \int_1^\infty \frac{2}{(2-3x)^{3/2}}\,dx = \lim_{t\to\infty}\int_1^t \frac{2}{(2-3x)^{3/2}}\,dx$$
$$= \lim_{t\to\infty}\frac{4}{3}(2-3x)^{-1/2}|_1^t = \lim_{t\to\infty}\frac{4}{3}((2-3t)^{-1/2}-(-1)^{-1/2})$$

At this point we see the problem that we are trying to take the square root of a negative number and stop.

$$\text{(b)} \int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx.$$

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx$$
$$= \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}|_0^t = \frac{1}{2} - \lim_{t\to\infty}\frac{1}{2}e^{-t^2} = \frac{1}{2}.$$

Same process for the other half, with value $-\frac{1}{2}$, so it is convergent with value 0.

II. Find the average value of the function on the given interval

$$\text{(a) Avg} = \frac{1}{2}\int_{-1}^1 (x^2+1)^2\,dx = \frac{1}{2}(\frac{1}{5}x^5 + \frac{2}{3}x^3 + x)|_{-1}^1 = \frac{28}{15}.$$

$$\text{(b) Avg} = \frac{1}{4}\int_0^4 \frac{1}{\sqrt{t}}\,dt = \frac{1}{4}\lim_{s\to 0}\int_s^4 \frac{1}{\sqrt{t}}\,dt = \frac{1}{4}\lim_{s\to 0} 2\sqrt{t}|_2^4$$
$$= \frac{1}{4}\lim_{s\to 0} 4 - 2\sqrt{s} = 1.$$

III. For each of the following problems, develop the appropriate approximation to the desired value and then express the exact value as an integral. Simplify, but do NOT evaluate the resulting integral.

(a) Slice the region into horizontal rectangles with thickness Δy. The area of 1 rectangle located at the value y_i, is $((2-y_i^2)-(-y_i))\Delta y$. Thus the approximate area is $S_n = \sum_{i=1}^n ((2-y_i^2)-(-y_i))\Delta y$. And the Area is $\lim_{n\to\infty} S_n = \int_{-1}^2 (2-y^2+y)\,dy$.

(b) Slice the volume horizontally to get disks with thickness Δy. The circular disk cut at a height of y_i has radius $2y_i - y_i^2$, so the volume of 1 disk is $\pi(2y_i - y_i^2)^2\Delta y$. The approximate volume is then $S_n = \sum_{i=1}^n \pi(2y_i - y_i^2)^2\Delta y$. And the total volume is then $\lim_{n\to\infty} S_n = \int_0^2 \pi(2y-y^2)\,dy$.

(c) Slice the volume vertically to get washers with thickness Δx. The washer cut at position x_i has inner radius of 1 and outer radius of $1+\frac{1}{x_i}$, thus the volume of 1 washer is $\pi((1+\frac{1}{x_i})^2 - 1^2)\Delta x = \pi(\frac{2}{x_i} + \frac{1}{x_i{}^2})\Delta x$. The approximate volume is then $S_n = \sum_{i=1}^n = \pi(\frac{2}{x_i} + \frac{1}{x_i{}^2})\Delta x$. The total volume is then $\lim_{n\to\infty} S_n = \int_1^3 \pi(\frac{2}{x} + \frac{1}{x^2}\,dx$.

(d) Slice the curve into pieces corresponding to a change in t of Δt. The approximate length of one piece cut at $t = t_i$ is $\sqrt{\Delta y^2 + \Delta x^2}$ where $\Delta y \approx y'(t_i)\Delta t$ and $\Delta x \approx x'(t_i)\Delta t$. Since $x'(t) = -a\sin t + b\cos t$ and $y'(t) = a\cos t + b\sin t$, $\Delta y^2 + \Delta x^2$ simplifies to $(a^2+b^2)\Delta t^2$, so the length of one piece is $\sqrt{a^2+b^2}\Delta t$. The approximate length of the entire curve is $S_n = \sum_{i=1}^n \sqrt{a^2+b^2}\Delta t$ and the actual length is $\lim_{n\to\infty} S_n = \int_0^\pi \sqrt{a^2+b^2}\,dt$.