Math 300 – Induction Problems

C-1. For each of the following sums, calculate the value for $n = 1$, $n = 2$ and $n = 5$. Also, write out the last term of the sum when n is replaced by $n + 1$.

(a) $\sum_{k=1}^{n}(2k-1)$

(b) $\sum_{k=1}^{n} k^2$

(c) $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$

(d) $\sum_{k=1}^{n} \frac{k}{2^k}$

C-2. Rewrite each of the following statements in terms of $n \in \mathbf{N}$:

(a) The sum of 3 consecutive natural numbers is divisible by 3.

(b) The product of 2 consecutive natural numbers is divisible by 2.

(c) The product of 3 consecutive natural numbers is divisible by 6.

(d) The sum of the cubes of 3 consecutive natural numbers is divisible by 9.

For each of the following determine if it is true or not for all $n \in \mathbf{N}$. If it is true, prove it. If it is not true for all n, determine what values of n it is true for and prove that that is so.

P-1. $\sum_{k=1}^{n}(2k-1) = n^2$.

P-2. $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$.

P-3. $\sum_{k=1}^{n} k^3 = [n(n+1)/2]^2$.

P-4. $\sum_{k=1}^{n}(2k-1)^2 = n(4n^2-1)/3$.

P-5. $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$.

P-6. $\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$.

P-7. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

P-8. $\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$.

P-9. If $a \in \mathbf{R}$ and $a \neq 1$, then $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$.

P-10. If $a \in \mathbf{R}$ and $a \neq 0$, then $\sum_{k=1}^{n}(a-1)a^{-k} = 1 - (1/a^n)$.

P-11. $5^n + 15^n$ is divisible by 10.

P-12. $73^n - 9^n$ is divisible by 64.

P-13. $n^2 + n$ is divisible by 2.

P-14. $n^5 - n$ is divisible by 5.

P-15. $7^n + 13^n$ is divisible by 5.

P-16. $2^n + 1$ is divisible by 3.

P-17. $n < 2^n$.

P-18. $n + 4 < 3^n$.

P-19. $2^{n-1} \leq n!$.

P-20. $2^n + n! < (n+2)!$.

P-21. $n! \leq n^n$.

P-22. $n^2 \leq 2^n + 1$.

P-23. If $x \in \mathbf{R}$ and $0 < x < 1$ then $x^n > x^{n+1}$.

P-24. If $x \in \mathbf{R}$ and $x > 1$ then $x^n < x^{n+1}$.

P-25. $3^{2-n} \geq \frac{2}{n!}$.

P-26. $n^2 \leq 2^n$.

P-27. $\sum_{k=1}^{n} k \leq n! + 1$.

P-28. $\sum_{k=1}^{2^n-1} \frac{1}{k} > \frac{n}{2}$.

P-29. $\sum_{k=1}^{2^n-1} \frac{1}{k^2} \leq 2 - \frac{1}{n}$.

P-30. $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n}$.

P-31. $\sum_{k=1}^{n} \frac{(-1)^k}{k} < 0$.

P-32. $2^n + 3^n \leq 5^n$.

P-33. $2^n \leq n!$.

P-34. $2(n!) + 2 \leq (n+1)!$.

P-35. $2^n \leq n! + 2$.

P-36. $\sum_{k=1}^{n} \frac{1}{k^3} \leq \frac{13}{8} - \frac{1}{n}$.

Recursion and Binomial Formula Problems

C-1. For the following recursion formulas, write out the first 5 terms and then write out the general expression for x_n in terms of n (if possible).

(a) $x_0 = 1$, $x_n = 2x_{n-1}$

(b) $x_0 = 1$, $x_n = x_{n-1} + 2$

(c) $x_0 = 0$, $x_n = (n-1)x_{n-1}/n$

(d) $x_0 = 3$, $x_n = x_{n-1} + n$

(e) $x_0 = 1$, $x_1 = x_0$, $x_2 = x_0 + x_1$, $x_n = x_0 + x_1 + \ldots + x_{n-1}$, $n \geq 3$

(f) $x_0 = 1$, $x_1 = x_0$, $x_2 = x_0 \cdot x_1$, $x_n = x_0 \cdot x_1 \cdot \ldots \cdot x_{n-1}$, $n \geq 3$

(g) $x_0 = 3$, $x_n = \frac{1}{2}x_{n-1} - 1$

(h) $x_0 = 1$, $x_n = 3x_{n-1} - 1$

C-2. Expand $(2-x)^4$ using the Binomial Formula.

C-3. Compute $\binom{9}{k}$ for $k = 0, 1, 2, \ldots, 9$.

C-4. Find the coefficient of x^3 in the expression $(x+2)^5$.

C-5. Find the coefficient of x^4 in the expression $(2x-3)^7$.

C-6. Find the expression of x^9 in the expression $(x^2+1)^{11}$.

C-7. Find the coefficient of x^9 in the expression $(5-7x)^{20}$.

C-8. Find the coefficient of x in the expression $(2x + \frac{1}{x})^6$.

Recursion and Binomial Formula Proofs (prove they hold for all $n \in \mathbf{N}$):

P-1. For each of C-1(a)-(h), prove that the formula you derived for x_n produces the same sequence as the recursion formula.

P-2. $\binom{n}{0} = \binom{n}{n} = 1$.

P-3. $\binom{n}{1} = \binom{n}{n-1} = n$.

P-4. If $n \geq 2$, $\binom{n}{2} = \binom{n}{n-2} = \frac{n(n-1)}{2}$.

P-5. $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

P-6. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

P-7. If $a, b \geq 0$ then $(a+b)^n \geq a^n + na^{n-1}b$.

P-8. $2 \leq (1 + \frac{1}{n})^n \leq 3$.

P-9. $2^n \geq n(n-1)(n-2)/6$.

P-10. $2(2^n - 1) = \binom{n+1}{1} + \ldots + \binom{n+1}{n}$.

P-11. $\sum_{k=1}^{n} \binom{n}{k} 2^k = 3^n - 1$.