Math 300 – Order Axiom and Absolute Value Problems

The following apply to an ordered field $\mathbb{F}$.

Definition: Let $A \subseteq \mathbb{F}$ with $A \neq \emptyset$.

The set A is *bounded above* if there is an $M \in \mathbb{F}$ such that $a \leq M$ for all $a \in A$. Such a M is called an *upper bound* of A.

A number $s \in \mathbb{F}$ is called a *supremum* of A (written $s = \sup A$) if it is an upper bound of A and for any upper bound M of A we have $s \leq M$. (Also called the *least upper bound* of A).

A number $m \in \mathbb{F}$ is called the *maximum* of A (written $m = \max A$) if it is a supremum of A and $m \in A$.

Examples:

$A = \{1, 2, 3, 4, 5, 6, 7\}$. Any number $M \geq 7$ is an upper bound. The supremum (and maximum) is 7.

$B = \{\frac{n-1}{n} \mid n \in \mathbb{N}\} = \{0, \frac{1}{2}, \frac{2}{3}, \ldots\}$. Any number $M \geq 1$ is an upper bound. The supremum is 1, but it is not the maximum.

$C = \{x \in \mathbb{R} \mid x^2 < 2\}$. For upper bounds, any number ≥ 2 clearly works. Looking carefully, we see that any $M \geq \sqrt{2}$ works as an upper bound. The supremum is $\sqrt{2}$. It is not the maximum.

The Completeness Axiom: The ordered field $\mathbb{F}$ is *complete* if $A \subseteq \mathbb{F}$, $A \neq \emptyset$, and A is bounded above, then A has a supremum in $\mathbb{F}$.

The real numbers $\mathbb{R}$ is the smallest complete ordered field. The rationals are not complete.

In all that follow assume all are elements of $\mathbb{R}$.

1. Find the supremum and maxiumum of the following sets (if they exist):

   $A = \{2n \mid n \in \mathbb{Z}\}$.

   $B = \{x \mid x^2 - 3x < 5\}$.

   $C = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$.

   $D = \{x \in \mathbb{Q} \mid x^2 < 5\}$.

   $E = \{1 + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$.

2. If $A \subseteq \mathbb{R}$ has a supremum, it is unique.

3. If $A \subseteq \mathbb{R}$ has a supremum s and $\epsilon > 0$ then $\exists a \in A$ such that $s - \epsilon < a$.

4. If A and B are nonempty, bounded subsets of $\mathbb{R}$ then

   (a) If $A \subseteq B$ then $\sup A \leq \sup B$.

   (b) If $A \cap B \neq \emptyset$ then $\sup(A \cap B) \leq \min(\sup A, \sup B)$.

   (c) If $C = \{a + b \mid a \in A, b \in B\}$ then $\sup C = \sup A + \sup B$.

5. (Archimedian Principle) The set $\mathbb{N}$ of natural numbers does not have an upper bound.

6. (variation of Archimmedian Principle) If $a, b \in \mathbb{R}$ and $a > 0$ and $b > 0$ then there exists a $n \in \mathbb{Z}$ such that $b < na$.

7. If $x \in \mathbb{R}$ and $x \notin \mathbb{Z}$ then there exists and $n \in \mathbb{Z}$ such that $n < x < n + 1$.

8. If $x \in \mathbb{R}$ and $n \in \mathbb{N}$ then there exists $p, q \in \mathbb{Z}$ such that $|x - p/q| < 1/n$.

9. If $x, y \in \mathbb{R}$ and $x < y$ then there exists a rational number r such that $x < r < y$.

   For nonempty $A \subseteq \mathbb{F}$ we can equivalently define

   a lower bound m by $m \leq a$ for all $a \in A$

   infinum $(\inf A)$ as a lower bound greater than or equal to all lower bounds

   minimum $(\min A)$ as an infinum which is contained in A

10. Determine the infinum and minimums (if the exist) of the sets in Problem 1.

11. State the infinum equivalent version of the Completeness Axiom and Problems 2, 3 and 4.

12. (Well-Ordering Principle) Every nonempty subset of $\mathbb{N}$ has a minimum.