# Math 300 – Set Theory Problems

For each statement prove it or find a counter-example. Unless otherwise stated, $A$, $B$ and $C$ are subsets of a universal set $U$.

1. $(A \cap B)^c = A^c \cup B^c$.
2. $(A \cup B)^c = A^c \cap B^c$.
3. $A \cap B = B \cap A$.
4. $A \cup B = B \cup A$.
5. $A \cap (B \cap C) = (A \cap B) \cap C$.
6. $A \cup (B \cup C) = (A \cup B) \cup C$.
7. $A \cap A = A$.
8. $A \cup A = A$.
9. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
10. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
11. $A \cup (A \cap B) = A$.
12. $A \cap (A \cup B) = A$.
13. $(A^c)^c = A$.
14. $A \cap U = A$.
15. $A \cup U = U$.

    Proofs involving the empty set ($\emptyset$) are different as you typically have to use proof by contradiction.

16. $U^c = \emptyset$.
17. $A \cap \emptyset = \emptyset$.
18. $A \cup \emptyset = A$.
19. $\emptyset^c = U$.
20. $A \cup A^c = U$.
21. $A \cap A^c = \emptyset$.

    In the next 4 problems you can prove them either by reducing to logic or by using Problems 1-21.

22. $(A \cup B) \cap (A \cup B^c) = A$.
23. $(((A \cap C) \cap B) \cup ((A \cap C) \cap B^c)) \cup (A \cap C)^c = U$.
24. $(A \cup C) \cap ((A \cap B) \cup (C^c \cap B)) = A \cap B$.
25. $B = A \cap (B \cup A^c)$.

**Define:** $A \setminus B$ by $x \in A \setminus B$ means $x \in A$ and $x \notin B$. (called the difference of $A$ and $B$ or the complement of $B$ in $A$)

26. $A \setminus B = A \cap B^c$.

27. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

28. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

29. $(A \setminus B) \cap B = \emptyset$.

30. $A \cap B = A \setminus (A \setminus B)$.

31. $((A \setminus B) \cup (B \setminus A))^c \cap A = A \cap B$.

32. $A \cap B = A \cap (B \setminus A^c)$.

33. $(A \cap B) \cup (A \setminus B) = A$.

34. $A \cup B = (A \setminus B) \cup (B \setminus A)$.

    **Define:** $A \triangle B = (A \setminus B) \cup (B \setminus A)$.) (called the symmetric difference of $A$ and $B$)

35. $A \triangle \emptyset = A$.

36. $A \triangle A = \emptyset$.

37. $A \triangle B = B \triangle A$.

38. $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

39. $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

40. $A \triangle B = (A \cup B) \setminus (A \cap B)$.

    To prove if-then statements, first assume the 'if' part (hypothesis, premise) and then use that to prove that the 'then' part (conclusion) is true.

41. If $A \subseteq B$ then $A \cup B = B$.

42. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

43. If $A \subseteq B$ and $B \subseteq C$ and $C \subseteq A$ then $A = B = C$.

44. $A \setminus (B \setminus A) = A$ if and only if $A \subseteq B$.

45. $A \cap B = B$ if and only if $B \subseteq A$.

46. (Cancellation Law) If $A \cap B = A \cap C$ and $A \cup B = A \cup C$ then $B = C$.

47. If $x \notin (A \cup B^c) \cup (A \setminus B)$ then $x \in B$.

48. $A \subseteq C$ if and only if $A \cup (B \cap C) = (A \cup B) \cap C$.

49. $B \subseteq A$ if and only if $B \setminus A = \emptyset$.

50. $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $A \subseteq C$.

The remaining problems involve different types of proof.

51. Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. If $x \notin D$ then $x \in B$.
52. If $A \cap C \subseteq B$ and $a \in C$, then $a \notin A \setminus B$.
53. Suppose $A \setminus B \subseteq C$. If $x \in A \setminus C$ then $x \in B$.
54. If $A \subseteq B$, $a \in A$ and $a \notin B \setminus C$, then $a \in C$.

**Define:** Two sets $A$ and $B$ are disjoint if $A \cap B = \emptyset$.

55. Suppose $A \subseteq C$ and $B$ and $C$ are disjoint. If $x \in A$ then $x \notin B$.
56. Suppose that $A \setminus B$ is disjoint from $C$ and $x \in A$. If $x \in C$ then $x \in B$.
57. If $A \setminus B \subseteq C$ then $A \setminus C \subseteq B$.
58. If $A \cap B = A$ then $A \subseteq B$.
59. If $A \subseteq B$ and $A$ and $C$ are disjoint, then $A \subseteq B \setminus C$.
60. $A \cap (B \setminus C) = (A \cap B) \setminus C$.
61. If $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.
62. Suppose $A \subseteq B$. For all sets $C$, $C \setminus B \subset C \setminus A$.
63. If $A \subseteq B$ and $A \not\subseteq C$ then $B \not\subseteq C$.
64. If $A \subseteq B \setminus C$ and $A \neq \emptyset$ then $B \not\subseteq C$.
65. If $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.
66. $A \setminus (B \setminus C) \subseteq (A \setminus B) \setminus C$.
67. $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.
68. $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.
69. $A \setminus (A \setminus B) = A \cap B$.
70. If $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$ then $A \subseteq B$.
71. If $A \triangle B \subseteq A$ then $B \subseteq A$.
72. $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.
73. $A \triangle B$ and $C$ are disjoint iff $A \cap C = B \cap C$.
74. $A \triangle B \subseteq C$ iff $A \cup C = B \cup C$.
75. $C \subseteq A \triangle B$ iff $C \subseteq A \cup B$ and $A \cap B \cap C = \emptyset$.
76. $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$.
77. $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$.
78. $(A \cup B) \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$.
79. $(A \triangle C) \cap (B \triangle C) \subseteq (A \cap B) \triangle C$.
80. If $A \setminus B \subseteq C$ and $A \not\subseteq C$ then $A \cap B \neq \emptyset$.

81. There is a unique set $A$ such that $A \cup B = B$ for all sets $B$.

82. If $A$ and $B$ are not disjoint, $A$ and $C$ are not disjoint and $A$ has exactly one element then $B$ and $C$ are not disjoint.

83. There is a unique set $X$ such that $A \triangle X A$ for all $A$.

84. For every set $A$ there is a unique set $B$ such that $A \triangle B = X$ ($X$ is the unique set from the previous problem).

85. Given any sets $A$ and $B$ there is a unique set $C$ such that $A \triangle C = B$.

86. For every $A$ there is a unique set $B \subseteq A$ such that for all $C \subseteq A$, $B \triangle C = A \setminus C$.

87. The following statements are equivalent

    (a) $(A \triangle C) \cap (B \triangle C) = \emptyset$

    (b) $A \cap B \subseteq C \subseteq A \cup B$

    (c) $A \triangle C \subseteq A \triangle B$.

88. (Disjoint Union Lemma) $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$ where any 2 of the sets on the right side are disjoint.

89. $A = (A \cap B) \cup (A \setminus B)$ and $(A \cap B) \cap (A \setminus B) = \emptyset$.