Extra Credit Extra Challenging Problems from Hutch

1. Determine, with proof, the number of ordered triples (A, B, C) of sets which have the property that $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A \cap B \cap C = \emptyset$.

2. The sequence of digits:

   $$123456789101112131415161718192021\ldots$$

   is obtained by writing the positive integers in order. if the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ since the 100th digit enters the sequence in the placement of the two digit integer 55. Find $f(1987)$.

3. If every point in the plane is painted one of three colors, do there necessarily exists two points of the same color exactly one inch apart?

   How about if three is replaced by nine? Justify your answers

4. How many primes among the positive integers, in base 10, are such that their digits are alternating 1s and 0s, beginning and ending with 1?

5. Find all positive integers within 250 of exactly 15 perfect squares.

6. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. the two players move alternately. a move consist of taking either (a) one bean from a heap, provided that at least two beans are left behind, or (b) a complete heap of two or three beans. The player who takes the last heap wins. To win the game do you want to go first or second? Give a winning strategy.

7. Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangle.

8. Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

9. Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find the number of subsets of $\{1, 2, 3, \ldots, n\}$ which are minimal selfish sets, that is , selfish sets none of whose proper subsets are selfish.

10. Players $1, 2, 3, \ldots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.