Math 300 – Intervals

Define the following notation for subsets (intervals) of $\mathbf{R}$:

$$\begin{aligned}
(a,b) &= \{x \in \mathbf{R} \mid a < x < b\}, && \text{for } a < b \\
(a,b] &= \{x \in \mathbf{R} \mid a < x \leq b\}, && \text{for } a < b \\
[a,b) &= \{x \in \mathbf{R} \mid a \leq x < b\}, && \text{for } a < b \\
[a,b] &= \{x \in \mathbf{R} \mid a \leq x \leq b\}, && \text{for } a \leq b
\end{aligned}$$

Comments:

- (a,b) is called an open interval and $[a,b]$ is called a closed interval. $[a,b)$ and $(a,b]$ are neither open nor closed. They are sometimes called half-open or half-closed.
- Note that none are defined for $a > b$ and the first 3 are not defined if $a = b$. The last one is defined for $a = b$ and $[a,a] = \{a\}$, i.e. the single point a.
- Note that when these definitions are expanded they have many parts. For example $x \in [1,2)$ means $((x > 1) \vee (x = 1)) \wedge (x < 2)$.

For any set $A \subseteq \mathbf{R}$ define the **reflection** of A to be the set

$$-A = \{x \in \mathbf{R} \mid -x \in A\}.$$

For each statement prove it or find a counter-example.

1. $\{x \in \mathbf{R} \mid 0 < 3x < 9\} = (0,3)$.
2. $\{x \in \mathbf{R} \mid 2 < 5x + 2 \leq 7\} = (0,1]$.
3. $\{x \in \mathbf{R} \mid 2 < 1 - x < 3\} = (-2,-1)$.
4. $\{x \in \mathbf{R} \mid -1 < 1 - 2x < 7\} = (-3,1]$.
5. $\{x \in \mathbf{R} \mid 0 < x^2 < 9\} = (-3,0) \cup (0,3)$.
6. $\{x \in \mathbf{R} \mid x^2 - 3x + 2 < 0\} = (1,2)$.
7. $\{x \in \mathbf{R} \mid x^2 > 9\} = (-\infty,-3) \cup (3,\infty)$. (since $x \in \mathbf{R}$ implies $-\infty < x < \infty$, $x \in (-\infty,a)$ means $x < a$ only and $x \in (a,\infty)$ means $x > a$.)
8. The intersection of two open intervals is either the empty set or an open interval.
9. The intersection of two closed intervals is either the empty set or a closed interval.
10. $-[1,2] = [-2,-1]$.
11. For $A \subseteq \mathbf{R}$, $x \in A$ if and only if $-x \in -A$.
12. For $A \subseteq \mathbf{R}$, $-(\mathbf{R} \setminus A) = \mathbf{R} \setminus (-A)$.
13. For $A, B \subseteq \mathbf{R}$, $-(A \cap B) = (-A) \cap (-B)$.
14. For $A, B \subseteq \mathbf{R}$, $-(A \cup B) = (-A) \cup (-B)$.
15. For $A, B \subseteq \mathbf{R}$, $-(A \setminus B) = (-A) \setminus (-B)$.