

Math 300 – Axiom Problems

1. Prove that if $ab = 0$ for some $a, b \in \mathbf{R}$, then $a = 0$ or $b = 0$.
2. Prove that if $e \in \mathbf{R}$ satisfies $a \cdot e = a$ for some nonzero $a \in \mathbf{R}$, then $e = 1$.
3. Prove that the multiplicative inverse of an element $a \in \mathbf{R}$, $a \neq 0$, is unique, i.e., if $b \in \mathbf{R}$ satisfies $ab = 1$, then $b = q(a)$.
4. Prove that $q(q(a)) = a$ for all $a \in \mathbf{R} \setminus \{0\}$.
5. Prove $n(0) = 0$.
6. Prove $n(a) = n(1) \cdot a$ for all $a \in \mathbf{R}$.
7. Prove $n(a + n(b)) = b + n(a)$ for all $a, b \in \mathbf{R}$.
8. Prove $n(1) \cdot n(1) = 1$.
9. Prove $n(a) \cdot b = n(a \cdot b)$ for all $a, b \in \mathbf{R}$.
10. Prove $n(a) \cdot n(b) = a \cdot b$ for all $a, b \in \mathbf{R}$.
11. Prove $q(1) = 1$.
12. Prove $q(a) \cdot q(b) = q(a \cdot b)$ for all $a, b \in \mathbf{R} \setminus \{0\}$.
13. Prove $q(a \cdot q(b)) = b \cdot q(a)$ for all $a, b \in \mathbf{R} \setminus \{0\}$.
14. Suppose $a, b \in \mathbf{R}$ and $a \neq 0$. Prove that $a \cdot x + b = 0$ has unique solution and express it using operators $n(\cdot)$ and $q(\cdot)$.
15. Suppose $a, b, c \in \mathbf{R}$. Prove that if $a \cdot b = a \cdot c$ then $a = 0$ or $b = c$.
16. Define x^2 as $x \cdot x$ and 2 as $1 + 1$. Then using the field axioms, prove that $(a + b)^2 = a^2 + 2ab + b^2$ for all $a, b \in \mathbf{R}$.
17. Define x^3 as $x^2 \cdot x$ and 3 as $2 + 1$. Then using the field axioms, prove that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ for all $a, b \in \mathbf{R}$.
18. Prove that $(a + b) \cdot q(c) = a \cdot q(c) + b \cdot q(c)$ for all $a, b, c \in \mathbf{R}$, $c \neq 0$.
19. Prove that $q(a) + q(b) = (a + b)q(ab)$ for all $a, b \in \mathbf{R} \setminus \{0\}$.