Math 300 – Takehome for Exam 3

Do one from each section. Do your own work.

1. Functions I

   (a) Let $f : \mathbf{R} \to \mathbf{R}$ defined by $f(z) = 2(z-5)+2$. Prove that f is one-to-one and onto.

   (b) Let $f : [0,5] \to [3,28]$ defined by $f(x) = (x-5)^2 + 3$. Prove that f is one-to-one and onto.

   (c) Find sets A and B such that $f : A \to B$ defined by $f(t) = (4t+5)/(t-11)$ is one-to-one and onto. Prove that it is so.

2. Functions II

   Def: A function $f : A \to B$ where $A \subseteq \mathbf{R}$ and $B \subseteq \mathbf{R}$ is *odd* if (1) for all $x \in A$, $-x \in A$, and (2) if $f(x) = -f(-x)$ for all $x \in A$. f is *even* if (1) holds and $f(x) = f(-x)$ for all $x \in A$.

   (a) Let m and b be real numbers. Prove that $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = mx + b$ is odd if and only if $b = 0$.

   (b) Let a, b and c be real numbers. Prove that $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = ax^2 + bx + c$ is even if and only if $b = 0$.

   (c) Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$. Prove that if f and g are odd then $f + g$ is odd and $f \cdot g$ is even. (Note: $(f+g)(x) = f(x) + g(x)$, $(f \cdot g)(x) = f(x) \cdot g(x)$)

   (d) Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$. Prove that if g is even then $f \circ g$ is even.

3. Induction

   (a) Prove that $73^n - 9^n$ is divisible by 64 for all $n \in \mathbf{N}$.

   (b) Prove that $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$ for all $n \in \mathbf{N}$.

   (c) Prove that if $a \in \mathbf{R}$ and $a \neq 0$, then $\sum_{k=1}^{n} (a-1)a^{-k} = 1 - (1/a^n)$ for all $n \in \mathbf{N}$.

   (d) Let $a, b \in \mathbf{N}$ with $a \neq b$. Prove that $a^n - b^n$ is divisible by $a - b$ for all $n \in \mathbf{N}$.

   (e) Prove that $2^n + 3^n \leq 5^n$ for all $n \in \mathbf{N}$.