Communcation over interference channels

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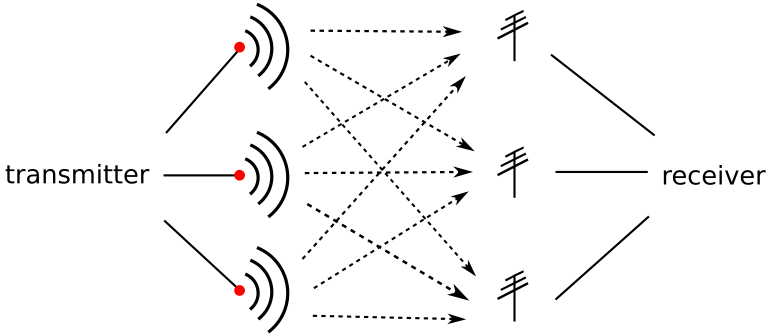
[1] work in progress with Guy Bresler and David Tse

Multiple-input multiple-output channel



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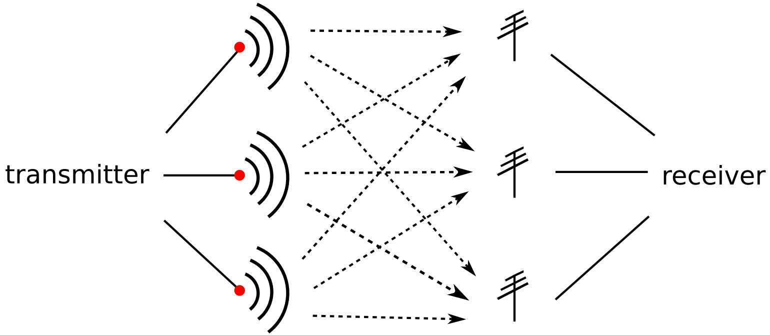

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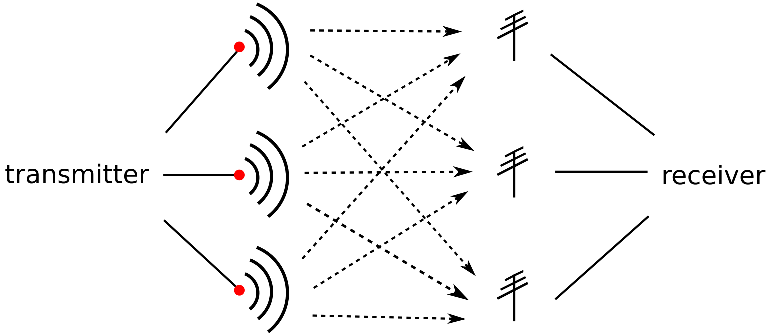

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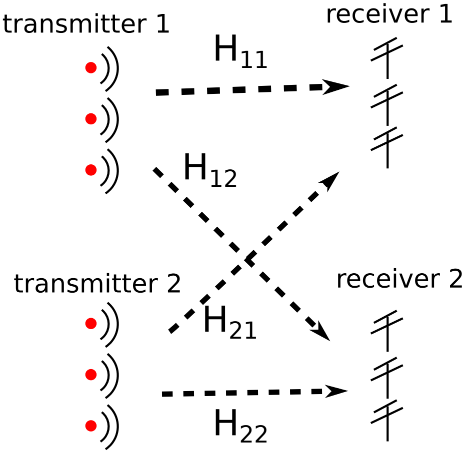
- If H is known and invertible, then the receiver can reconstruct the message v.

Multiple users of the same channel



- K transmitter-receiver pairs using the same channel.
- Determined by K^2 channel matrices H_{ij} of size $N \times N$.

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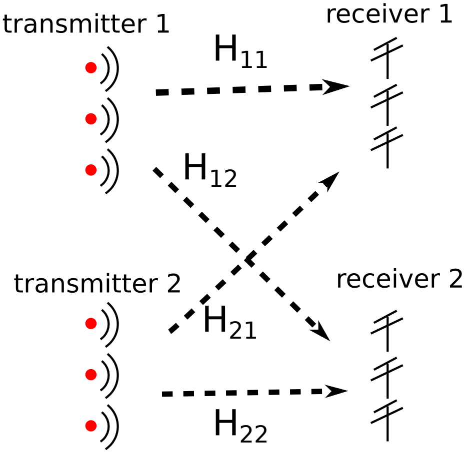

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- Reciever 1 only cares about transmitter 1's message, etc.

Strategies for interference alignment

- Each transmitter has a subspace $V_j \subset \mathbb{C}^N$ to transmit in.
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If each H_{ii} is generic, the second condition is satisfied automatically.

Questions

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- What is the information capacity of this channel?
- How to parametrize spaces of solution strategies?

Incidence correspondence

$$\left(\mathbb{C}^{N\times N}\right)^{K(K-1)} \times \prod_{i=1}^{K} \mathrm{Gr}(d_i, N) \times \mathrm{Gr}(N-d_i, N)$$

Subvariety of those

$$(H_{12}, \ldots, H_{K-1,K}, V_1, \ldots, V_K, U_1, \ldots, U_K)$$

such that

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Question

Is the projection onto $\left(\mathbb{C}^{N\times N}\right)^{K(K-1)}$ *surjective?*

Existence of solutions

Theorem

Assume that $d = d_1 = \cdots = d_K$ and $K \geq 3$. Then a generic set of channel matrices has a solution if and only if

$$2N \geq d(K+1).$$

If so, the dimension of the solution variety is

$$dK\big(2N - d(K+1)\big)$$

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For non-constant d_i, we have the necessary conditions:

$$d_i + d_j \leq N \quad \text{for all } i, j$$

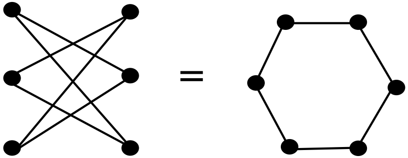
$$\sum_{i \in S} 2d_i(N - d_i) \geq \sum_{i \neq j \in S} d_i d_j \quad \text{for all subsets } S \subset \{1, \dots, K\}$$

$K = 3$

The threshold case for feasibility is

$$(d_1, d_2, d_3) = (d, d, N - d),$$

where $d_1 \leq N/2$.



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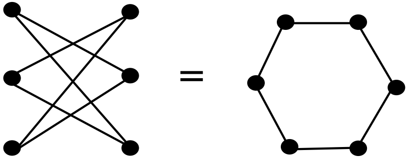

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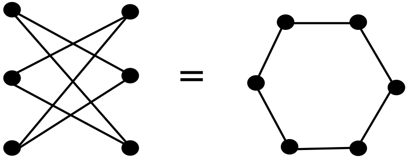

- After change of coordinates, can assume that all but one channel matrix is the identity.
- For dimension reasons, inclusions become equalities:

$$V_1 = U_3 = V_2 \subset U_1 = V_3 = U_2 \supset H_{21} V_1$$

An eigenvector-like problem

Given generic $N \times N$ matrix H, find

- $V \subset \mathbb{C}^N$, subspace of dimension d
- $U \subset \mathbb{C}^N$, subspace of dimension $e = N - d$

such that

$$V \subset U \qquad \text{and} \qquad HV \subset U$$

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- More generally, for $d = e > 1$, take $V = U$ to be spanned by d eigenvectors. In particular, $\binom{N}{d}$ solutions.
- For $e > d$, variety of solutions of dimension

$$\big(N - (e - d)\big)(e - d)$$

Parametrizing the solution variety

Recall: Want to find V, U such that $V \subset U$ and $HV \subset U$.

- Set $\ell := \left\lfloor \dfrac{d}{e-d} \right\rfloor$
- Choose $S \subset \mathbb{C}^N$ of dimension $d - \ell(e-d)$.

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Structure of whole variety seems complicated: when $e = d+1$, then it is the toric variety for the Minkowski sum of hypersimplices $\Delta_{e,N} + \Delta_{N,d}$.

Numbers of solutions

Return to

$$K \geq 3 \qquad d = d_1 = \ldots = d_k$$

Zero-dimensional when $N = \frac{d(K+1)}{2}$. The number of solutions is:

	K				
d	3	4	5	6	7
1	2	-	216	-	1,975,560
2	6	3700	388,407,960		
3	20	-		-	
4	70				
$\vdots$	$\vdots$				
d	$\binom{2d}{d}$				

Number of solutions when $d = 1$

Assume $d = d_1 = \cdots d_K = 1$ and $2N = K + 1$.
Degenerate each H_{ij} to a rank 1 matrix:

$$H_{ij} V_j \subset U_j \qquad \Longleftrightarrow \qquad V_j \subset \ker H_{ij} \;\text{ or }\; U_j \supset \operatorname{im} H_{ij}$$

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Theorem

Number of solutions = number of balanced orientations of the graph G
G has edges $t_j - s_i$ whenever $i \neq j$. Balanced orientation means that

$$\textit{in degree}(v) = \textit{out degree}(v) = \frac{K-1}{2}$$

for all vertices v of G.

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- Can we parametrize the solution variety in more cases?
- What if the receivers and transmitters have different numbers of antennas?
- What if the channel matrices have the form

$$H_{ij} = \begin{bmatrix} \tilde{H}_{ij} & 0 & \cdots & 0 \\ 0 & \tilde{H}_{ij} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{H}_{ij} \end{bmatrix}?$$