Secant varieties of Segre-Veronese varieties

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[1] joint with Daniel Erman and Luke Oeding

Tensor rank for matrices

U, V: finite dimensional vector spaces

$x \in U \otimes V$

The rank of x is the smallest integer r such that x can be written

$$x = u_1 \otimes v_1 + \cdots + u_r \otimes v_r \text{ where } u_i \in U, v_i \in V.$$

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- The rank is the same if we pass to a bigger field.
- The set of possible decompositions is a homogeneous space.

Partially symmetric tensors

U: m-dimensional $\mathbb{C}$-vector space

V: n-dimensional $\mathbb{C}$-vector space

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The rank can jump both down and up for special tensors.

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The border rank of x is the smallest integer r such that x can be approximated arbitrarily closely by expressions of the form:

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The set of such decompositions will in general be a finite set of points, possibly defined over a larger field than x.

Equations for bounded border rank

When $U = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, we can write

$$x = e_1 \otimes A + e_2 \otimes B$$

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Theorem

The ideal of partially symmetric tensors whose border rank is at most r is generated by the $(r+1) \times (r+1)$-minors of the block matrix

$$\begin{pmatrix} A & B \end{pmatrix}$$

Equations for small border rank

When $U = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$, we can write

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Theorem (C.-Erman-Oeding 2010)

If $r \leq 5$, the ideal of tensors whose border rank is at most r is generated by the $(r+1) \times (r+1)$-minors and $(2r+2) \times (2r+2)$-Pfaffians respectively of

$$\begin{pmatrix} A & B & C \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & A & -B \\ -A & 0 & C \\ B & -C & 0 \end{pmatrix}$$

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Remark

The $n = 4$, $r = 5$ case is due to Emil Toeplitz in 1869.

Outline of the proof

- Assume $n = r$ and A is the identity matrix. Then

$$\begin{pmatrix} 0 & I & -B \\ -I & 0 & C \\ B & -C & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & BC - CB \end{pmatrix}$$

The $2r + 2$-Pfaffians of this matrix include the entries of the commutator $BC - CB$, which is a prime, Gorenstein ideal, defining the variety of commuting symmetric matrices.

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- Arbitrary n. Here the minors come in.

Unifying framework for these equations

Given decomposable $u \otimes v \otimes v \in U \otimes S^2 V$, we have linear map

$$\begin{aligned}\psi_{j,u\otimes v\otimes v} \colon V^* \otimes \bigwedge^{j} U &\to V \otimes \bigwedge^{j+1} U \\ v^* \otimes (u_1' \wedge \cdots \wedge u_j') &\mapsto \langle v^*, v \rangle v \otimes u_1' \wedge \cdots \wedge u_j' \wedge u\end{aligned}$$

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If U is 3-dimensional,

- The $j = 0$ and $j = 2$ cases give the rectangular matrix
- The $j = 1$ case gives the skew-symmetric square matrix

Robust testing of determinantal equations

Let

$$\sigma_1 \geq \cdots \geq \sigma_n \qquad \text{and} \qquad \sigma'_1 \geq \cdots \geq \sigma'_{3n}$$

be the singular values of

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$$\gamma_r^2 = \sum_{i=r+1}^{n} \sigma_i^2 \qquad\qquad \delta_r^2 = \sum_{i=2r+1}^{3n} (\sigma'_i)^2$$

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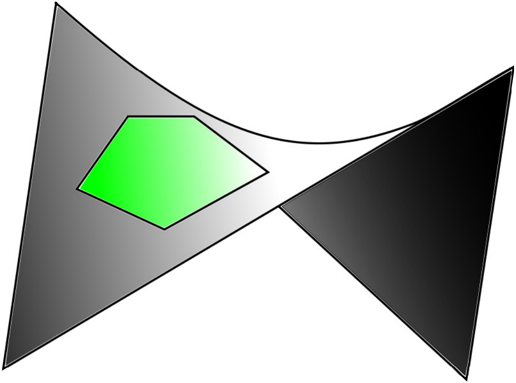
are continuous, non-negative functions which are both zero if and only if the tensor has border rank at most r.

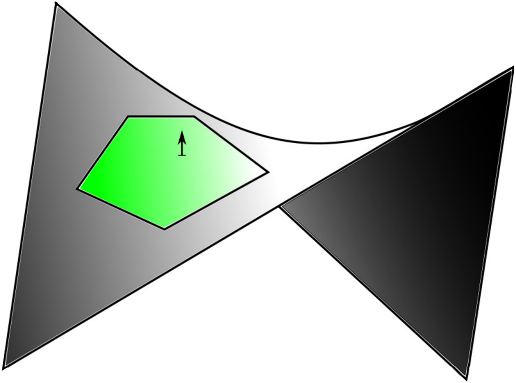
Bounded real rank is a semi-algebraic set

Tensors with real border rank at most r characterized by same equalities, but additional inequalities



Bounded real rank is a semi-algebraic set

Equalities are more important than inequalities for detecting deviations



Thank you