Construction of the Lindström valuation of an algebraic matroid

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Algebraic matroids

Given $K \subset L = K(x_1, \ldots, x_n)$ a field extension, algebraic matroid of $K \subset L$ is:

- Independent sets are sets I such that $\{x_i \mid i \in I\}$ are algebraically independent
- Bases are the maximal independent sets (called transcendence bases in field theory)
- Circuits are the minimal dependent sets
- Rank function $\mathrm{rk}(S)$ is the transcendence degree of the extension $K \subset K(x_i \mid i \in S)$
- Geometrically: If $L = \mathrm{Frac}(K[x_1, \ldots, x_n]/I)$, then $rk(S)$ is the dimension of the projection defined by $K[x_i \mid i \in S] \cap I$

Algebraic matroids are hard!

Differentials: Linearizing algebraic relations

- If $K \subset L = K(x_1, \ldots, x_n)$ is a field extension, the vector space of differentials $\Omega_{L/K}$ is a L-vector space whose dimension is the transcendence degree of $K \subset L$, generated by elements $dx_1, \ldots, dx_n$
- Geometrically: If $L = \mathrm{Frac}(K[x_1, \ldots, x_n]/I)$, then $\Omega_{L/K}$ is the dual to the tangent space of $V(I)$ at the generic point
- If K has characteristic 0, the algebraic matroid of $K \subset L$ is the same as the linear matroid of the differentials $\Omega_{L/K}$ with vectors $dx_1, \ldots, dx_n$
- On the other hand, if K has characteristic p, and $L = \mathrm{Frac}\, K[x_1, x_2]/\langle x_1^p - x_2 \rangle$, then:
  - dx_1 is non-zero, $dx_2 = 0$, so only basis of linear matroid of $\Omega_{L/K}$ is $\{1\}$
  - Bases of the algebraic matroid of $K \subset L$ are $\{1\}$ and $\{2\}$

Frobenius function $x \mapsto x^p$ is weird in characteristic p

Tropicalization of a vector space

- Set-up: k field with valuation $\mathrm{val}\colon k^\times \twoheadrightarrow \Gamma \subset \mathbb{R}$, V a k-vector space, $x_1, \ldots, x_n \in V$
- Given: $w = (w_1, \ldots, w_n) \in \Gamma^n$
- Scale: $t_1^{w_1} x_1, \ldots, t_n^{w_n} x_n$, where $t^{w_i} \in k$, $\mathrm{val}(t^{w_i}) = w_i$
- Generate: R-submodule of V generated by $t^{w_1} x_1, \ldots, t^{w_n} x_n$, where R is the valuation ring of k
- Reduce: tensor with R/mR to get $\mathrm{in}_w(V)$, where m is the maximal ideal of R
- Tropicalization: $\mathrm{Trop}(V) \cap \Gamma^n$ is the set of $w \in \Gamma^n$ such that reductions

$$\overline{t^{w_1} x_1}, \ldots, \overline{t^{w_n} x_n} \in \mathrm{in}_w(V)$$

  are all non-zero
- The tropicalization is equivalent to the valuated matroid of V

Rough idea

"Tropicalization" for fields:

- Scaling $\Longrightarrow$ Frobenius
- Reduction $\Longrightarrow$ Differentials

“Tropicalization” of field extensions

- Set-up: K field of char. p, $L = K(x_1, \dots, x_n)$
- Given: $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$
- Scale: $F^{-w_1}x_1, \dots, F^{-w_n}x_n$, where F is Frobenius: $Fx = x^p$, in $\tilde{L} = \bigcup_l K(x_1^{1/p^l}, \dots, x_n^{1/p^l})$
- Generate: $K(F^{-w}x) := K(F^{-w_1}x_1, \dots, F^{-w_n}x_n)$
- Reduce: Vector space of differentials $\Omega_{K(F^{-w}x)/K}$ generated by differentials $dF^{-w_1}x_1, \dots, dF^{-w_n}x_n$
- Tropicalization: $\mathrm{Trop}(L/K) \cap \mathbb{Z}^n$ is the set of $w \in \mathbb{Z}^n$ such that differentials $dF^{-w_1}x_1, \dots, dF^{-w_n}x_n$ are all non-zero
- $\mathrm{Trop}(L/K)$ is the tropicalization of a unique valuated matroid, called the Lindström valuated matroid of $K \subset L$ (Bollen-Draisma-Pendavingh)

Local and global structure

If V is a k-vector space, $x_1, \ldots, x_n \in V$, then $\mathrm{Trop}(V)$ is

- Globally: The recession fan of $\mathrm{Trop}(V)$ is equivalent to the linear (non-valuated) matroid of $x_1, \ldots, x_n \in V$
- Locally: At $w \in \Gamma^n$, the link of $\mathrm{Trop}(V)$ is equivalent to the linear matroid of $\mathrm{in}_w(V)$

If $K \subset L = K(x_1, \ldots, x_n)$ is a field extension, then $\mathrm{Trop}(L/K)$ is

- Globally: The recession fan of $\mathrm{Trop}(L/K)$ is equivalent to the algebraic matroid of $K \subset L$
- Locally: The link of $w \in \mathbb{Z}^n$ is equivalent to the linear matroid of $\Omega_{K(F^{-w}x)/K}$

A bridge example: monomials

- A: a $d \times n$ integer matrix
- $L = K(z_1, \ldots, z_d)$, $x_i = z_1^{A_{1i}} \cdots z_d^{A_{di}}$
- The Lindström valuated matroid of $K \subset L$ is the same as the valuated matroid of the columns of A in $\mathbb{Q}^d$ with the p-adic valuation
- Prior example of $K[x_1, x_2]/\langle x_1^p - x_2 \rangle$ is a monomial example with

$$A = \begin{bmatrix} 1 & p \end{bmatrix}$$

Circuits of the Lindström matroid

As before: $K \subset L = K(x_1, \ldots, x_n)$

- If C is a circuit of the algebraic matroid of $K \subset L$, then $\{x_i \mid i \in C\}$ is a minimal dependent set, and there exists a polynomial relation $f_C \in K[x_i \mid i \in C]$, unique up to scaling
- Write
$$f_C = \sum_{u \in J_C} c_u x_1^{u_1} \cdots x_n^{u_n}$$
where $J_C \subset \mathbb{Z}^n_{\geq 0}$, $u_i = 0$ if $i \notin C$, and $c_u \neq 0$
- Define:
$$\mathbf{C}(f_C) = \big(\ldots, \min\{\mathrm{val}_p\, u_i \mid u \in J_C\}, \ldots \big) \in (\mathbb{Z} \cup \infty)^n$$
where val_p is the p-adic valuation
- The valuated circuits of the Lindström valuation are the vectors
$$\mathbf{C}(f_C) + \lambda \mathbb{1}$$
as C ranges over circuits of the algebraic matroid, and $\lambda \in \mathbb{Z}$

Valuation of the Lindström matroid

As before: $K \subset L = K(x_1, \ldots, x_n)$

- Let B be a basis of the algebraic matroid, meaning a maximal independent set of variables
- The extension $K(x_i \mid i \in B) \subset L$ can be uniquely factored as

  $$K(x_i \mid i \in B) \subset K(x_i \mid i \in B)^{\mathrm{sep}} \subset L$$

  where the first extension is separable (roughly: like characteristic 0) and the second is purely inseparable (defined by taking pth roots)
- The Lindström valuation is:

  $$v(B) = \log_p [L : K(x_i \mid i \in B)^{\mathrm{sep}}] \in \mathbb{Z}_{\geq 0}$$

Cocircuits of the Lindström valuation

As before: $K \subset L = K(x_1, \ldots, x_n)$

- Let H be a hyperplane of the algebraic matroid, meaning a maximal set with $\mathrm{rk}(H) = \mathrm{rk}(\{1, \ldots, n\}) - 1$
- Define:

$$\mathbf{C}^{\mathrm{co}}(H) = \big(\ldots, \log_p [L : K(x_i \mid i \in H \cup \{i\})^{\mathrm{sep}}], \ldots \big)$$

- The valuated cocircuits of the Lindström valuation are:

$$\mathbf{C}^{\mathrm{co}}(H) + \lambda \mathbb{1}$$

as H ranges over the hyperplanes, and $\lambda \in \mathbb{Z}$