Tropical complexes

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Overview

Analogy between algebraic curves and finite graphs. For example, Baker's specialization lemma:

$$h^0(X, \mathcal{O}(D)) - 1 \leq r(\operatorname{Trop} D)$$

Main goal: generalize the specialization inequality to higher dimensions.

Tropical complexes: higher-dimensional graphs

An n-dimensional tropical complex is a finite Δ-complex Γ with simplices of dimension at most n, together with integers $a(v, F)$ for every $(n-1)$-dimensional face (facet) F and vertex $v \in F$, such that Γ satisfies the following two conditions:

First, for each facet F,

$$\sum_{v \in F} a(v, F) = -\#\{n\text{-dimensional faces containing } F\}$$

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Remark

A 1-dimensional tropical complex is just a graph because the extra data is forced to be $a(v, v) = -\deg(v)$.

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Second, for any $(n-2)$-dimensional face G, we form the symmetric matrix M whose rows and columns are indexed by facets containing G with

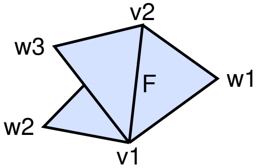
$$M_{FF'} = \begin{cases} a(F \setminus G, F) & \text{if } F = F' \\ \#\{\text{faces containing both } F \text{ and } F'\} & \text{if } F \neq F' \end{cases}$$

and we require all such M to have exactly one positive eigenvalue.

Local charts

A tropical complex locally has a map to a real vector space.

F: $(n-1)$-dimensional simplex in a tropical complex Γ



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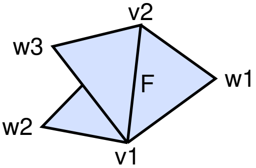

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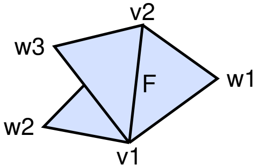

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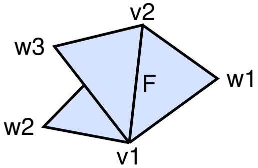
V_F: quotient vector space $\mathbb{R}^{n+d}/(a(v_1, F), \ldots, a(v_n, F), 1, \ldots, 1)$

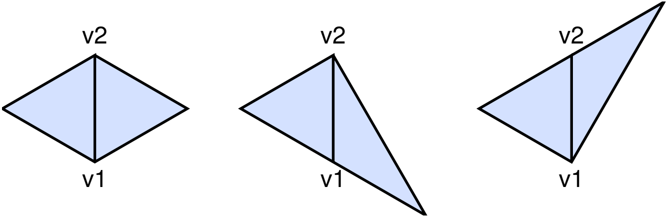
ϕ_F: linear map $N(F) \to V_F$ sending v_i and w_j to images of ith and $(n+i)$th unit vectors respectively.



Example: two triangles meeting along an edge

$n = d = 2$.

Γ consists of two triangles sharing a common edge F.



$a_1 = a_2 = -1$ $\quad$ $a_1 = -2, a_2 = 0$ $\quad$ $a_1 = 0, a_2 = -2$

where a_i is shorthand for $a(v_i, F)$.

Linear and piecewise linear functions

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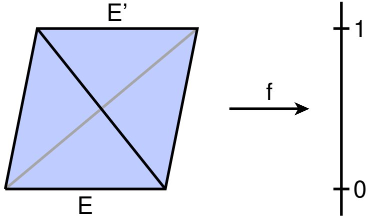
- A piecewise linear function f has an associated divisor, which is a formal sum of $(n-1)$-dimensional polyhedra supported where the function is not linear.

Example: Tetrahedron

Γ is the boundary of a tetrahedron with all $a(v, F) = -1$.



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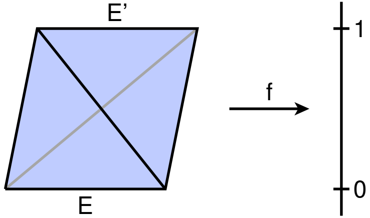

The divisor of f is $2[E] - 2[E']$.

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Why $n-3$? Roughly, Weil divisors are balanced, which is a condition in dimension $n-2$.

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Definition

Let Γ be a tropical complex and D a Weil divisor on it. Define $h^0(\Gamma, D) \in [0, \infty]$ to be the smallest integer k such that there exist k rational points $x_1, \ldots, x_k$ in Γ such that D is not linearly equivalent to any effective divisor containing all the x_i.

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- For any $I \subset [n]$, any component of $\cap_{i \in I} C_i$ is called a stratum.
- The dual complex is a Δ-complex with one k-dimensional cell for each $(n-k)$-dimensional stratum. The faces of a cell correspond to strata containing a given one.

Tropical complex of a semistable degeneration

We assume that the open strata (the difference of one stratum minus all strata strictly contained in it) are affine. Then, dual complex is also a tropical complex:

- $a(v, F)$ is the self-intersection of the curve corresponding to F in the surface corresponding to $F \setminus v$, the face of F not containing v.

Specialization inequality

If D is a divisor on the general fiber of $\mathfrak{X}$, then define

$$\mathrm{Trop}(D) = \sum_{F \in \Gamma^{(n-1)}} (\overline{D} \cdot C_F)[F],$$

where $\overline{D}$ is the closure of D in $\mathfrak{X}$, and C_F is the 1-dimensional stratum corresponding to the facet F.

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Theorem

Under our hypotheses on $\mathfrak{X}$ (or somewhat weaker), for any divisor on the general fiber of $\mathfrak{X}$,

$$h^0(X, \mathcal{O}(D)) \leq h^0(\Gamma, \operatorname{Trop} D)$$

Summary of other results

Comparison theorem:

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- Tropical Hodge index theorem.
- Tropical Noether's formula:

$$12\chi(\Gamma) = \int_\Gamma c_1^2 + c_2$$