On dual complexes of degenerations

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Degenerations

R: rank 1 valuation ring

K: fraction field of R

val: valuation on K

$\mathfrak{X}$: flat, proper scheme over $\operatorname{Spec} R$

n: relative dimension of $\mathfrak{X}$

Definition

We say that $\mathfrak{X}$ is a (strictly semistable) degeneration over R if locally $\mathfrak{X}$ has an étale morphism over R to $\operatorname{Spec} R[x_0, \dots, x_n]/\langle x_0 \cdots x_m - \pi \rangle$ for some $0 \leq m \leq n$ and some $\pi \in R$ with $0 < \operatorname{val}(\pi) < \infty$.

A stratum of codimension m is a connected subset of $\mathfrak{X}$ consisting of points with an étale morphism to the origin in $\operatorname{Spec}[x_0, \dots, x_n]/\langle x_0 \cdots x_m - \pi \rangle$.

Dual complexes

Definition

The dual complex Δ of a degeneration $\mathfrak{X}$ is a Δ-complex which consists of an m-dimensional simplex s for each codimension m stratum C_s of $\mathfrak{X}$. The faces u of s correspond to strata C_u such that $\overline{C}_u \supset C_s$.

Example

If $g \in R[w,x,y,z]$ is a generic polynomial of degree d and $\ell_1, \ldots, \ell_d$ are generic linear forms in $R[w,x,y,z]$, then a small resolution of

$$\operatorname{Proj} R[w,x,y,z]/\langle g - \pi \ell_1 \cdots \ell_d \rangle$$

is a strictly semistable degeneration of dimension 2. Its dual complex is the complete simplicial complex of dimension 2 on d vertices.

The dual complex Δ is homotopy equivalent to the Berkovich analytification $(\mathfrak{X}_K)^{\mathrm{an}}$.

Things I'm not doing

In many contexts, people either:

- Assume that R is discretely valued and π generates the maximal ideal of R ($\mathfrak{X}$ is regular).
- Identify stratum $\mathrm{Spec}[x_0, \ldots, x_n]/\langle x_0 \cdots x_m - \pi \rangle$ with m-dimensional simplex scaled by $\mathrm{val}(\pi)$.

Dual complexes of curves

Fact

Any finite, connected graph is the dual complex of a 1*-dimensional degeneration* $\mathfrak{X}$ *over any complete discrete valuation ring.*

Dual complexes of surfaces

There exist degenerations with dual complexes homeomorphic to the following:

surface	dual complex
K3	sphere S^2
Abelian surface	torus $S^1 \times S^1$
Enriques surface	projective plane RP^2
bielliptic surface	Klein bottle $(S^1 \times S^1)/(\mathbb{Z}/2)$

Theorem (C)

Given a 2-dimensional degeneration whose dual complex Δ is homeomorphic to a topological surface, then $\chi(\Delta) \geq 0$, i.e. it is one of the homeomorphism types listed above.

Conjecture

Homeomorphic be strengthened to homotopy equivalent in this theorem.

Hyperbolic manifold with fins and ornaments

Definition

A hyperbolic manifold with fins and ornaments is a Δ-complex Δ with subcomplexes $\Sigma, F_1, \ldots, F_k, O$ such that:

- $\Delta = \Sigma \cup F_1 \cup \cdots \cup F_k \cup O$.
- Σ *is homeomorphic to a connected 2-dimensional topological manifold with* $\chi(\Sigma) < 0$.
- F_i *is contractible and* $F_i \cap \Sigma$ *is a path.*
- *For* $i > j$, $F_i \cap F_j$ *is a subset of the endpoints of the path* $F_i \cap \Sigma$.
- $O \cap (\Sigma \cup F_1 \cup \cdots \cup F_k)$ *is finite.*

Theorem (C)

There does not exist a 2-dimensional degeneration whose dual complex Δ is a hyperbolic manifold with fins and ornaments.

Tropical exponential sequence

Let Δ be the dual complex of a degeneration of surfaces. Using certain intersection numbers the special fibers, we can construct a sheaf of affine linear functions $\mathcal{A}$ on Δ such that:

- In codimension 1, this sheaf looks like affine linear functions with integral slopes on (tropical curve) $\times \mathbb{R}$.
- Affine linear functions are defined to be continuous functions which are affine linear in codimension 1.

Let $\mathcal{D}$ be the quotient sheaf $\mathcal{A}/\mathbb{R}$ so that we have a long exact sequence:

$$\to H^0(\Delta, \mathcal{D}) \xrightarrow{\delta} H^1(\Delta, \mathbb{R}) \to H^1(\Delta, \mathcal{A}) \to H^1(\Delta, \mathcal{D}) \to$$

analogous to the exponential sequence on a complex projective variety Y:

$$\to H^1(Y, \mathbb{Z}) \to H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y^*) \to H^2(Y, \mathbb{Z}) \to$$

Ingredients for proof of theorem

$$\to H^0(\Delta, \mathcal{D}) \xrightarrow{\delta} H^1(\Delta, \mathbb{R}) \to H^1(\Delta, \mathcal{A}) \to H^1(\Delta, \mathcal{D}) \to$$

Proposition (C)

Possibly after adding more fins, $\mathbb{R}(\operatorname{im}\delta)$ has codimension at most 1 in $H^1(\Delta, \mathbb{R})$.

Proposition (C)

If Δ is a (hyperbolic) manifold with fins, then

$$H^0(\Delta, \mathcal{D}) \to H^0(U, \mathcal{D}) \equiv \mathbb{Z}^2$$

is an isomorphism.

Putting these results together, $H^1(\Delta, \mathbb{R}) \leq 3$, which implies $\chi(\Delta) \geq -1$.