The Gröbner stratification of a tropical variety

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Gröbner basics

$$I \subset K[x_0, \ldots, x_n] : \text{ homogeneous ideal}$$
$$w \in \mathbb{R}^n : \text{ weight vector}$$

For a polynomial:

$$f = \sum_{\mathbf{i}} a_i x_0^{i_0} \cdots x_n^{i_n} \in K[x_0, \ldots, x_n]$$

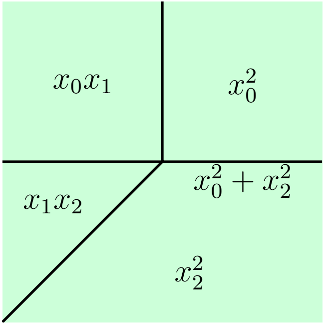
we define the w-weight of the monomial $x_0^{i_0} \cdots x_n^{i_n}$ to be $w_1 i_1 + \cdots + w_n i_n$

$$\mathrm{in}_w(f) = \sum_{\mathbf{i} \text{ with minimal weight}} a_i x_0^{i_0} \cdots x_n^{i_n}$$
$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle$$

Gröbner complex

The Gröbner complex of an ideal I is a finite polyhedral fan in $\mathbb{R}^n$ such that $w, w' \in \mathbb{R}^n$ are in the relative interior of the same cell if and only if

$$\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$$

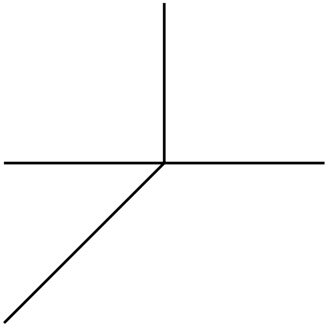


Gröbner complex of $\langle x_0^2 + x_0x_1 + x_1x_2 + x_2^2 \rangle$

Tropical variety

The tropicalization $\mathrm{Trop}(I)$ of an ideal I is the closed union of cones of the Gröbner complex defined by:

$$\begin{aligned} w \in \mathrm{Trop}(I) \quad &\Leftrightarrow \quad \mathrm{in}_w(I) \text{ does not contain any monomial} \\ &\Leftrightarrow \quad \mathrm{in}_w(I) \cdot K[x_0^{\pm}, \ldots, x_n^{\pm}] \neq \langle 1 \rangle \end{aligned}$$



Tropicalization of $\langle x_0^2 + x_0 x_1 + x_1 x_2 + x_2^2 \rangle$

What does the tropicalization tell us?

- If I is prime of dimension d (and not containing a monomial), then the maximal cells of $\mathrm{Trop}(I)$ have dimension d.
- Valuations of solutions of I over valued extensions of K.
- Instructions for a partial resolution of singularities in a toric variety.

Gröbner complex in tropical geometry

The tropicalization can be defined without the Gröbner complex, e.g.:

- image of Berkovich analytification,
- valuations of solutions to I over valued extensions of K,
- intersection of tropical hypersurfaces.

Question

What does the Gröbner complex tell us beyond the tropicalization?

Homogeneous polynomials versus Laurent polynomials

Technical difference: Gröbner complex constructed in $K[x_0, \ldots, x_n]$ and tropicalization constructed in the Laurent polynomial ring $K[x_0^{\pm}, \ldots, x_n^{\pm}]$.

- For example, a non-linear monomial automorphism of $K[x_0^{\pm}, \ldots, x_n^{\pm}]$ such as:

  $$x_1 \mapsto x_1^2 x_2, x_2 \mapsto x_1^3 x_2$$

  acts by linear transformation on the tropicalization, but changes the Gröbner complex drastically.
- Many of the properties we look for in $\mathrm{Trop}(I)$ only depend on $I \cdot K[x_0^{\pm}, \ldots, x_n^{\pm}]$

The Gröbner stratification

The Gröbner stratification of an ideal $I \subset K[x_1^{\pm}, \dots, x_n^{\pm}]$ is stratification of $\mathbb{R}^n$ such that $w, w' \in \mathbb{R}^n$ are in the same locally closed strata if and only if

$$\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$$

and there exists a continuous path from w to w' which also has the same initial ideal (i.e. strata are forced to be connected).

- The Gröbner stratification is a coarsening of the Gröbner complex, but the cells are not necessarily convex.
- The tropical variety is the complement of the maximal strata.

Refined question

Question

Is the Gröbner stratification complete determined by the tropicalization?

But first: Add a valuation to the mix

Suppose that K has a valuation $\mathrm{val}\colon K^* \to \mathbb{R}$., with a section denoted $\nu \mapsto t^\nu$.

Then, we define the *w*-weight of a term $a_i x_0^{i_0} \cdots x_n^{i_n}$ to be

$$\mathrm{val}(a_i) + w_1 i_1 + \cdots + w_n i_n.$$

$$\mathrm{in}_w \Big(\sum_i a_i x_0^{i_0} \cdots x_n^{i_n} \Big) = \sum_{i \text{ with minimal weight}} \overline{a_i} x_0^{i_0} \cdots x_n^{i_n}$$

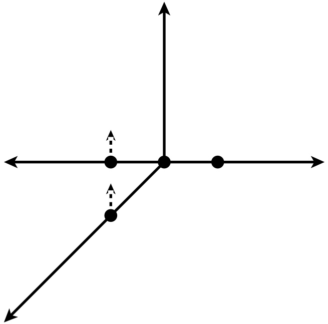
where $\overline{a_i}$ is the reduction of $a_i t^{-\mathrm{val}(a_i)}$ in the residue field of K.

Same as before except: Gröbner complex, tropicalization, etc. are no longer invariant under positive rescalings of w.

The main example

Let $\pi \in K \ni 1/3$ be an element of valuation 1 and let let $I \subset K[x^{\pm}, y^{\pm}, z^{\pm}]$ be generated by the 2×2 minors of:

$$\begin{bmatrix} x-1 & \pi(y-1) & \pi(y-1-\pi z) \\ \pi(y-1) & \pi(y-1-\pi z) & x-1-\pi \end{bmatrix}, \tag{1}$$



Gröbner stratification of I sees ideal-theoretic information beyond the cycle-theoretic information of the tropicalization.

The example is as small as possible

Theorem (C)

The Gröbner stratification of the tropicalization of a prime ideal I is as coarse as possible if either:

- *I is a principal, or*
- $\mathrm{Trop}(I)$ *is locally matroidal.*

For the second case, the essential point is to show that $\mathrm{in}_w(I)$ is reduced.