Tropical complexes

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Tropical curves: an overview

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Goal: Extend this analogy to higher dimensions.

Hypersurfaces in Fano toric varieties

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The boundary of P^o (as a simplicial complex) is called the dual complex of the degeneration.

Tropical complexes

An n-dimensional tropical complex is a Δ-complex Γ of pure dimension n, together with integers $a(v, F)$ for every $(n-1)$-dimensional face (facet) F and vertex $v \in F$, such that Γ satisfies the following two conditions:
First, for each face F,

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Remark

A 1-dimensional tropical complex is just a graph because the extra data is forced to be $a(v, v) = -\deg(v)$.

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Second, for any $(n-2)$-dimensional face G, we form the symmetric matrix M whose rows and columns are indexed by facets containing G with

$$M_{FF'} = \begin{cases} a(F \setminus G, F) & \text{if } F = F' \\ \#\{\text{faces containing both } F \text{ and } F'\} & \text{if } F \neq F' \end{cases}$$

and we require all such M to have exactly one positive eigenvalue.

Local embeddings

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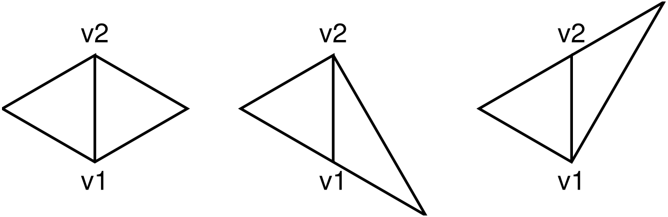
A continuous $\mathbb{R}$-valued function on Γ is linear if on each $N(F)^o$ it is the composition of ϕ_F followed by an affine linear function with integral slopes.

Example: two triangles meeting along an edge

$n = d = 2$.

Γ is two triangles sharing a common edge F.



$a_1 = a_2 = -1$ $\quad$ $a_1 = -2, a_2 = 0$ $\quad$ $a_1 = 0, a_2 = -2$

where a_i is shorthand for $a(v_i, F)$.

Divisors

Definition

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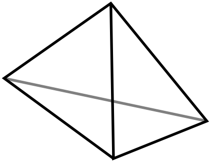
Definition

A divisor is a formal sum of $(n-1)$-dimensional polyhedra which is locally the divisor of a piecewise linear function.

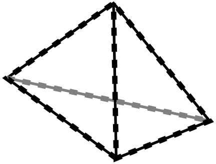
Definition

Two divisors are linearly equivalent if their difference is the divisor of a (global) piecewise linear function.

Example: The 1-skeleton of a tetrahedron



Γ is the boundary of a tetrahedron, with all $a(v, F) = -1$.



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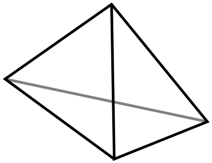

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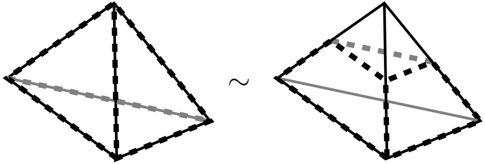

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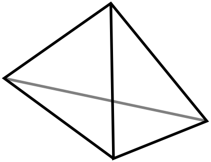

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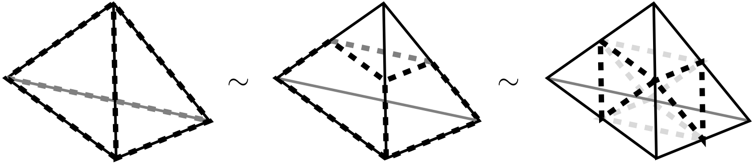

Intersections on surfaces

Let D and D' be two divisors on a 2-dimensional tropical complex. Locally, write D as the divisor of a piecewise linear function f. Define the product of D and D' as a formal sum of points of D' for which p has multiplicity:

$$\sum_{E\colon \text{edge of } D', E \ni p} (\text{outgoing slope of } f \text{ along } E)(\text{multiplicity of } E \text{ in } D')$$

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Proposition

This itersection product is well-defined and symmetric. The degree of the resulting 0-cycle is invariant under linear equivalence of both D and D'.

Hodge index theorem

Theorem

Let Γ be 2-dimensional tropical complex such that the link of every vertex is connected. If H is a divisor on Γ such that $H^2 > 0$ and D a divisor such that $H \cdot D = 0$, then $D^2 < 0$.

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Conjecture

On any 2-dimensional tropical complex where the link of every vertex is connected, there exists a divisor H such that $H^2 > 0$.