TREE COMPACTIFICATIONS

DUSTIN CARTWRIGHT AND DIANE MACLAGAN

Abstract. We construct a class of combinatorially described compactifications of $M_{0,n}$, each admitting a map from $\overline{M}_{0,n}$. Up to normalization, our compactifications include the weighted moduli spaces of Hassett, the modular compactifications of Smyth, and many of the GIT constructions of Giansiracusa-Jensen-Moon. Our compactifications are constructed as closures inside toric varieties, and are always proper algebraic varieties, but not always projective.

1. Introduction

The moduli space $M_{0,n}$ of rational curves with n marked points is a smooth affine variety of dimension $n-3$. In addition to the well-known Grothendieck-Deligne-Mumford-Knudsen compactification $\overline{M}_{0,n}$, a number of alternative compactifications have been constructed using weighted points [Has03], modular considerations [Smy13], and GIT of Chow varieties [GJM13]. In this paper, we give a new family of compactifications of $M_{0,n}$, which includes many of the previously listed examples. Our compactifications are parametrized by combinatorial data, which we call an admissible collection of trees.

Theorem 1.1. *For any admissible collection of trees $\mathcal{F}$, there exists a compactification $M_{\mathcal{F}}$ of $M_{0,n}$, together with a birational morphism $\phi_{\mathcal{F}}\colon \overline{M}_{0,n} \to M_{\mathcal{F}}$.*

An admissible collection is a set of trees with n labeled leaves satisfying combinatorial conditions listed in Definition 2.3 below. The properties of $M_{\mathcal{F}}$ and the behavior of the birational morphism $\phi_{\mathcal{F}}$ are determined combinatorially by the collection $\mathcal{F}$. For example, recall that $\overline{M}_{0,n}$ has a stratification by locally closed subsets $\overline{M}_{0,n}^{\tau}$ indexed by trees τ whose leaves are labeled by $\{1,\dots,n\}$. The strata for which $\phi_{\mathcal{F}}(\overline{M}_{0,n}^{\tau})$ is a point are those such that some contraction of τ is in the admissible collection $\mathcal{F}$. More generally, the admissible collection determines the image of any stratum of $\overline{M}_{0,n}$, and these images form a stratification of $M_{\mathcal{F}}$, indexed by a class of trees called $\mathcal{F}$-stable. See Propositions 4.5 and 4.8 for details.

The construction of the compactification $M_{\mathcal{F}}$ is by taking the closure of $M_{0,n}$ in a toric variety determined by the collection of trees $\mathcal{F}$. Recall that $\overline{M}_{0,n}$ can be realized as a tropical compactification, meaning that $M_{0,n}$ embeds in $(\mathbb{G}_m)^{\binom{n-1}{2}-1}$ and $\overline{M}_{0,n}$ is isomorphic to the closure of $M_{0,n}$ in a toric variety defined by a fan supported on the tropicalization of $M_{0,n}$ in this embedding [Tev07, GM10]. Given an admissible collection of trees $\mathcal{F}$, we construct a fan $\Sigma_{\mathcal{F}}$ and define $M_{\mathcal{F}}$ to be the closure of $M_{0,n}$ in the toric variety corresponding to $\Sigma_{\mathcal{F}}$. Except in the case of $\overline{M}_{0,n}$, $M_{\mathcal{F}}$ is not a tropical compactification, and in general the fan $\Sigma_{\mathcal{F}}$ is not equidimensional. Morphisms between different compactifications $M_{\mathcal{F}}$, including the map $\phi_{\mathcal{F}}$ from Theorem 1.1, are obtained from toric maps between the corresponding toric varieties.

One virtue of this construction is that it encompasses most compactifications of $M_{0,n}$ known in the literature, at least up to normalization. For example, modular compactifications, as defined by Smyth [Smy13], parametrize possibly degenerate curves, which are allowed to have coinciding points or worse than nodal singularities, depending on the parameters. In genus 0, Smyth classified these modular degenerations in terms of the combinatorial data of an extremal assignment Z. An extremal assignment can be translated into an admissible collection of trees $\mathcal{F}_Z$.

Theorem 1.2. *Any modular compactification of $M_{0,n}$ in the sense of Smyth is isomorphic to a tree compactification $M_{\mathcal{F}_Z}$, up to normalization. More precisely, there exists a birational and bijective morphism from $M_{\mathcal{F}_Z}$ to the modular compactification $M_{0,n}(Z)$.*

We are not aware of any cases when the morphism $M_{\mathcal{F}_Z} \to M_{0,n}(Z)$ from Theorem 1.2 is not an isomorphism. While Smyth proves that his modular compactifications are represented by algebraic spaces [Smy13], Theorem 1.2 shows that, at least after normalization, they are in fact algebraic varieties.

In a different direction, Giansiracusa, Jensen, and Moon generalized Kapranov's construction of $\overline{M}_{0,n}$ from [Kap93] to get a family of compactifications of $M_{0,n}$ coming from Chow varieties using GIT [GJM13]. We give an explicit translation of the parameters from their construction into an admissible collection of trees. For generic GIT parameters, their construction is isomorphic to a Smyth modular compactification, and thus to a tree compactification, up to normalization. Our specification of the tree compactification extends to the walls and other special GIT parameters, and for these we also get a regular morphism from the normalization of the tree compactification to the GIT quotient, as described in Theorem 6.3.

Smooth tree compactifications are interesting from both combinatorial and modular perspectives. Combinatorially, smooth tree compactifications can be characterized in terms of the trees allowed in admissible collection, and such admissible collections have an equivalent description in terms of what we call combinatorial weight data. Moreover, all smooth tree compactifications have a modular interpretation, parametrizing genus 0 curves with at worst nodal singularities and marked points which are allowed to coincide under circumstances controlled by the combinatorics. Smooth tree compactifications include the Hassett weighted moduli spaces [Has03], and can be related by blow-ups along smooth subvarieties in an analogous manner.

While tree compactifications and modular compactifications are always proper algebraic varieties, they are not always projective. This is in contrast to the GIT construction of [GJM13] which only yields projective varieties. The existence of proper, non-projective smooth modular compactifications was first noted in [MSvAX18, Ex. 11.1].

The rest of this paper is structured as follows. In Section 2 we introduce admissible collections of trees and describe their combinatorics. Section 3 contains the construction of the compactification from an admissible collection of trees. In Section 4, we establish a stratification of the tree compactification, generalizing the stratification by combinatorial type on $\overline{M}_{0,n}$. Sections 5 and 6 study the relationship with the compactifications due to Smyth and Giansiracusa-Jensen-Moon, respectively. Finally, Section 7 is devoted to smooth tree compactifications.

Acknowledgements. The ideas that led to this paper began when both authors were visiting the Institut Mittag-Leffler, which we thank for its excellent working conditions. We thank Arend Bayer for helpful conversations about the spaces studied in Section 7 and their relation to Smyth's work. The first author was partially supported by NSF grant DMS-1103856L and NSA grant H98230-16-1-0019. The second author was partially supported by EPSRC grant EP/I008071/1 and a Leverhulme Trust International Academic Fellowship.

2. Admissible collections of trees

In this section we construct the compactification $M_{\mathcal{F}}$ associated to an admissible collection of trees. We begin by giving some preliminary combinatorial definitions related to trees with labeled leaves, which will be needed for the definition of an admissible collection of trees. Throughout this section, n will be a fixed positive integer, denoting the number of leaves in the tree, as well as the number of marked points for the moduli space $\overline{M}_{0,n}$. We work over an arbitrary field $\Bbbk$.

Notation 2.1. By a *tree*, we will always mean an acyclic graph with n leaves labeled by the set $[n] = \{1, \ldots, n\}$ and without 2-valent vertices. The vertices that are not leaves will be called *internal vertices.* A *leaf edge* is an edge adjacent to a leaf and an *internal edge* is an edge that is not a leaf edge. A *contraction* of a tree is the tree obtained by contracting zero or more internal edges. An *expansion* of a tree is the reverse of a contraction.

A tree is *trivalent* if all internal vertices have valency three. The *star tree* with n leaves is the tree with no internal edges and only one non-leaf vertex.

Definition 2.2. By a *partition* of $[n]$, we mean an unordered collection of one or more subsets, called parts, $P_1, \ldots, P_m \subset [n]$, which are disjoint and whose union is $[n]$. A *refinement* of a partition with parts $P_1, \ldots, P_m$ is another partition with parts $P'_1, \ldots, P'_s$ where each P'_i with $1 \leq i \leq s$ is contained in some P_j for $1 \leq j \leq m$. In that case we call the first partition a *coarsening* of the second.

The *partition* $\operatorname{part}(v)$ of $[n]$ induced by an internal vertex v of a tree T is the set of leaf labels of the connected components of the graph obtained by deleting v from T.

The terms from Notation 2.1 and Definition 2.2 are illustrated by the trees shown in Figure 1.

Definition 2.3. An *admissible collection* of trees $\mathcal{F}$ is a set of trees with n labeled leaves such that:

(1) The star tree with n leaves is not in $\mathcal{F}$.
(2) For any trivalent tree T, there is exactly one contraction of T that is in $\mathcal{F}$.
(3) If v is a vertex of a tree T in $\mathcal{F}$, then any tree T' that is trivalent except at a vertex v' with $\operatorname{part}(v')$ a coarsening of $\operatorname{part}(v)$ has a contraction in $\mathcal{F}$.

If P is a partition of $[n]$ that coarsens $\operatorname{part}(v)$ for some vertex v of some tree $T \in \mathcal{F}$, then we say that P is an *admissible partition* of the collection $\mathcal{F}$.

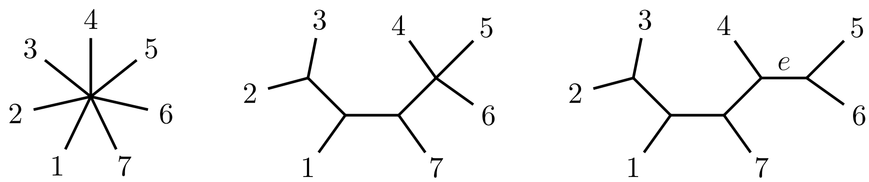

FIGURE 1. All three of the pictured trees have $n = 7$ labeled leaves. The tree on the left is the star tree, with a unique internal vertex. In the center is an expansion of the star tree to a tree with 4 internal vertices, and 3 internal edges. This tree is not trivalent because of the rightmost internal vertex. The partition induced by this non-trivalent vertex is $\{1237, 4, 5, 6\}$. On the right is a trivalent expansion of the middle tree at this non-trivalent vertex, creating the edge e. Conversely, the middle tree is the contraction of the edge e on the right tree.

The collection of all trivalent trees is an admissible collection, whose associated compactification is $\overline{M}_{0,n}$. Our goal in the next two sections is to construct a compactification of $M_{0,n}$ from an admissible collection of trees.

Recall that the boundary of $\overline{M}_{0,n}$ has a stratification according to the the number of components of the parametrized curves, or equivalently the number of internal vertices and internal edges of the dual graph. The strata corresponding to dual graphs with $n-4$ internal edges are one-dimensional, and their closures are known as F-curves. The dual graphs are then trees that are trivalent except for a single 4-valent vertex.

Example 2.4. When $n = 4$, the set of all trivalent trees is the only admissible collection.

Consider now the case $n = 5$. Since an admissible collection $\mathcal{F}$ does not contain the star tree by the first axiom of Definition 2.3, it must consist only of trivalent trees, which will have two internal edges, and non-trivalent trees with exactly one internal edge. Moreover, by the second axiom, the trivalent trees in $\mathcal{F}$ must be exactly the set of trivalent trees which are not expansions of the non-trivalent trees in $\mathcal{F}$. Thus, the set of non-trivalent trees in $\mathcal{F}$ determines the admissible collection. Again by the second axiom, the non-trivalent trees must have the property that no two of them share a common refinement.

The trees with 5 leaves and exactly one internal edge correspond to the 1-dimensional strata of $\overline{M}_{0,5}$, whose closures are the F-curves. A set of trees, each with one internal edge and such that no pair has a common refinement corresponds to a collection of (-1)-curves in $\overline{M}_{0,5}$ which are pairwise non-intersecting.

The third axiom of Definition 2.3 is vacuous for $n = 5$.

Example 2.5. Let $n \geq 5$ and fix an F-curve C corresponding to a tree τ with 4-valent vertex v. Let $\mathcal{F}$ consist of the trees τ' such that τ' is either: a trivalent tree except for a 4-valent v' for which $\mathrm{part}(v) = \mathrm{part}(v')$; or, a trivalent tree such that no contraction of τ' is in the previous case. One can check that $\mathcal{F}$ defined in this way is an admissible collection of trees. Note that the trees in the first case of our description correspond to F-curves in the same numerical equivalence class as C.

In Example 2.5, the non-trivalent vertices that occurred in a tree in $\mathcal{F}$ were characterized by their partitions. In fact, trees in any admissible collections are characterized by their partitions, as follows.

Lemma 2.6. *Let $\mathcal{F}$ be an admissible collection of trees. If T is a tree such that for every vertex v of T, $\operatorname{part}(v)$ is an admissible partition of $\mathcal{F}$, then a contraction of T lies in $\mathcal{F}$.*

Proof. Let T be as in the statement, and choose τ to be a trivalent expansion of T. Now fix a vertex v of T. Let T_v be the tree which is a contraction of τ and trivalent except at a single vertex which has partition equal to $\operatorname{part}(v)$. Then by the third axiom of an admissible collection, there exists a tree $T'_v \in \mathcal{F}$ that is a contraction of T_v. However, as we vary v over the vertices of T, the trees T'_v are all contractions of the trivalent tree τ, so they must all be equal to a single tree T'. Since every edge of τ that is contracted in T is contracted in some T_v, this means that T' is a contraction of T. □

Lemma 2.6 shows that an admissible collection of trees determines and is determined by the collection of its admissible partitions. We can go further and give an axiomatization equivalent to an admissible collection of trees via the following definition and Propositions 2.8 and 2.10 below.

Definition 2.7. A collection $\mathcal{P}$ of partitions of $[n]$ is an *admissible collection of partitions* if it satisfies the following four properties:

(1) If π is a partition of $[n]$ with at most three parts, then $\pi \in \mathcal{P}$;
(2) The refined partition $\{1\}, \{2\}, \ldots, \{n\}$ of $[n]$ is not in $\mathcal{P}$;
(3) If $\pi \in \mathcal{P}$, and π' is a coarsening of π, then $\pi' \in \mathcal{P}$;
(4) Suppose π is a partition of $[n]$ with l parts, π' is the coarsening of π obtained by replacing the first r parts by a single part, and π'' is the coarsening of π obtained by replacing the last s parts by a single part, where $r + s < l$. If $\pi', \pi'' \in \mathcal{P}$, then $\pi \in \mathcal{P}$.

Proposition 2.8. *If $\mathcal{F}$ is an admissible collection of trees, then the set of admissible partitions of $\mathcal{F}$ in the sense of Definition 2.3 is an admissible collection of partitions as in Definition 2.7.*

Proof. Let $\mathcal{F}$ be an admissible collection of trees. Since all trivalent trees have a contraction in $\mathcal{F}$, all three-part partitions are admissible for $\mathcal{F}$. Since the star tree is not in $\mathcal{F}$, the refined partition is not admissible for $\mathcal{F}$. This shows the first two conditions for an admissible collection of partitions. The third is immediate from the definition of an admissible partition for $\mathcal{F}$.

Suppose now that π is a partition of $[n]$ with l parts, and π', π'' are partitions obtained from π by replacing some parts by a single part as in condition 4, with π', π'' admissible for $\mathcal{F}$. Fix a tree T of the form shown in Figure 2, where $I_1, \ldots, I_l$ are the parts of the partition π, and every vertex except the vertices of e and e' are trivalent. Let T' be the tree obtained by contracting the edge e'. The vertex v obtained by contracting e' has $\operatorname{part}(v) = \pi'$, so by the third condition on admissible collections of trees we have that T' has a contraction in $\mathcal{F}$. Let T'' be the tree obtained by contracting the edge e. The vertex v obtained by contracting e has $\operatorname{part}(v) = \pi''$,

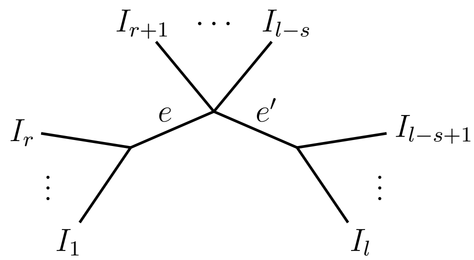

FIGURE 2. The tree T used in the proof of Proposition 2.8 The labels $I_1, \ldots, I_l$ refer to the labels occurring in each of the subtrees.

so again T'' has a contraction in $\mathcal{F}$. As these are both contractions of any trivalent expansion of T, these contractions must coincide, so in particular the tree obtained from T by contracting both e and e' has a contraction in $\mathcal{F}$. This means that π is an admissible partition for $\mathcal{F}$, so condition 4 is satisfied. □

The following definition is used in the proof of Proposition 2.10.

Definition 2.9. The *refined partition* of an edge e in a tree T is the partition of the vertex formed by contracting e. In other words, the refined partition of e is the common refinement of the partitions at the endpoints of e.

Proposition 2.10. *Given an admissible collection of partitions $\mathcal{P}$, set $\mathcal{T} = \{T : T$ is a tree with $\operatorname{part}(v) \in \mathcal{P}$ for all $v \in V(T)\}$. Let $\mathcal{F}$ be the set of trees T from $\mathcal{T}$ such that no proper contraction of T is in $\mathcal{T}$. Then $\mathcal{F}$ is an admissible collection of trees.*

Proof. Since the refined partition $\{1\}, \ldots, \{n\}$ is not in $\mathcal{P}$, the star tree is not in $\mathcal{T}$, and thus not in $\mathcal{F}$. Since all trivalent trees are in $\mathcal{T}$, they all have contractions in $\mathcal{F}$. Suppose now that T is a trivalent tree, and T', T'' are two different contractions of T that both lie in $\mathcal{F}$. Since $T' \neq T''$ there is some edge e of T that is contracted in T' but not in T''. The image v' of e in T' has $\operatorname{part}(v') \in \mathcal{P}$, so the refined partition of e, which coarsens $\operatorname{part}(v')$, is also in $\mathcal{P}$. Let e'' be the image of e in T''.

We next show that the refined partition of e'' is also in $\mathcal{P}$. This will imply that the tree obtained by contracting the edge e'' in T'' is thus also in $\mathcal{T}$, which contradicts $T'' \in \mathcal{F}$. From this it follows that there is a unique contraction of T in $\mathcal{F}$.

The proof is by induction on the number of edges contracted in T to form T'', with the base case and induction step being identical. Suppose that $\tilde{T} \in \mathcal{T}$, and $\tilde{e}$ is an edge of $\tilde{T}$ with refined partition in $\mathcal{P}$. Let $\tilde{T}'$ be the tree obtained by contracting $\tilde{e}$. If $\tilde{e}'$ is another edge in $\tilde{T}$, with refined partition in $\mathcal{P}$, then it suffices to show that the refined partition of the image of $\tilde{e}'$ in $\tilde{T}'$ lies in $\mathcal{P}$. If $\tilde{e}$ and $\tilde{e}'$ do not share a vertex, then the refined partition of $\tilde{e}'$ is the same in $\tilde{T}$ and $\tilde{T}'$, so we only need to consider the case that they share a vertex v. We are then again in the situation of Figure 2. Since the refined partitions of both $\tilde{e}$ and $\tilde{e}'$ lie in $\mathcal{P}$, the refined partition of $\tilde{e}$ in $\tilde{T}'$ is also in $\mathcal{P}$ by Part 4 of Definition 2.7.

Finally, the third condition for admissible collections of trees follows because if v is a vertex of a tree $T \in \mathcal{F}$ we have $\operatorname{part}(v) \in \mathcal{P}$, so $\operatorname{part}(v') \in \mathcal{P}$ whenever $\operatorname{part}(v')$ coarsens $\operatorname{part}(v)$. Since all three-part partitions are in $\mathcal{P}$, it follows that if T' is a tree that is trivalent except at a vertex v' with $\text{part}(v')$ a coarsening of $\text{part}(v)$, then $T' \in \mathcal{T}$, and so T' has a contraction in $\mathcal{F}$. $\square$

3. Construction of the compactifications

Given an admissible collection, in this section we construct the tree compactification as the closure of $M_{0,n}$ in a toric variety whose fan is determined by the admissible collection.

Let $\mathbb{R}^{\binom{n}{2}}$ denote the vector space with basis consisting of the vectors $\mathbf{e}_{ij} = \mathbf{e}_{ji}$ for $1 \leq i, j \leq n$. Set $N_\mathbb{R}$ to be the quotient of $\mathbb{R}^{\binom{n}{2}}$ by the n-dimensional vector space generated by $\{\sum_{j \neq i} \mathbf{e}_{ij} \colon 1 \leq i \leq n\}$. By abuse of notation, we also use $\mathbf{e}_{ij}$ to denote its image in $N_\mathbb{R}$. We set $N \subset N_\mathbb{R}$ to be the lattice generated by the vectors $\mathbf{e}_{ij}$.

For any subset I of $[n]$ such that $2 \leq |I| \leq n-2$, we define

$$\mathbf{v}_I = \sum_{i<j \in I} \mathbf{e}_{ij} = -\frac{1}{2} \sum_{i \in I, j \notin I} \mathbf{e}_{ij} = \sum_{i<j \notin I} \mathbf{e}_{ij} \in N. \tag{1}$$

As a consequence, $\mathbf{v}_I = \mathbf{v}_{I^c}$, where I^c denotes the complement $[n] \setminus I$. In other words, $\mathbf{v}_I$ only depends on the partition consisting of the unordered sets I, I^c, although we continue to denote it by $\mathbf{v}_I$ for notational convenience.

Since $N_\mathbb{R}$ is a quotient of $\mathbb{R}^{\binom{n}{2}}$, its dual, which we denote by $M_\mathbb{R}$ is a codimension-n subspace of $\mathbb{R}^{\binom{n}{2}}$. We denote by $\mathbf{f}_{ij} = \mathbf{f}_{ji} \in \mathbb{R}^{\binom{n}{2}}$, for $1 \leq i \neq j \leq n$, the dual to $\mathbf{e}_{ij}$. The dual lattice $M \subset M_\mathbb{R}$ is generated by elements of the form $\mathbf{f}_{ij;kl} = \mathbf{f}_{ik} + \mathbf{f}_{jl} - \mathbf{f}_{il} - \mathbf{f}_{jk}$. For any set I,

$$\mathbf{f}_{ij;kl} \cdot \mathbf{v}_I = \begin{cases} 1 & \text{if } I \cap \{i,j,k,l\} = \{i,k\} \text{ or } \{j,l\} \\ -1 & \text{if } I \cap \{i,j,k,l\} = \{i,l\} \text{ or } \{j,k\} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Definition 3.1. Let T be a tree. A *split* of T at an internal edge e is the partition consisting of the sets of labels on the two connected components formed by deleting e from T. In addition, we define a *division* of T to be any partition of $[n]$ into two parts that coarsens the partition $\text{part}(v)$ for some vertex v of T.

Every split at an edge e is also a division at either of the endpoints of e. However, if T is not trivalent, then there is a division at any non-trivalent vertex that is not a split of T. Nonetheless, every division of T is a split of some expansion of T and conversely, every split of any expansion of T is a division of T.

Definition 3.2. Given a tree T, we define the cone C_T in $N_\mathbb{R}$ to be the set of non-negative linear combinations of the vectors $\mathbf{v}_I$ as I ranges over all divisions of T.

Our goal for the rest of the section is to show that from an admissible collection $\mathcal{F}$, the set of cones $\{C_T : T \in \mathcal{F}\}$ form the maximal cones of a polyhedral fan defining a toric variety. We begin by constructing hyperplanes that will be used to separate distinct cones. If T is a tree and $I \subset [n]$, we define the *restriction* of T to I, denoted by $T|_I$, to be the unique connected subgraph of T whose leaves consist of the vertices of T labeled by I. We consider this subgraph to be a tree as in Definition 2.1 by merging edges incident to 2-valent vertices.

Proposition 3.3. *The function $\mathbf{f}_{ij;kl}$ is non-negative on a cone C_T if and only if the restriction $T|_{\{i,j,k,l\}}$ is trivalent with i and l not in the same split of the restriction.*

Proof. Suppose that the restriction of T to these labels is trivalent with i and l not in the same split. It suffices to check non-negativity on the rays $\mathbf{v}_I$ for I a division of T. Since I is a division of T, it is not possible for I to contain i and l but neither of j and k or vice versa, so, by (2), $\mathbf{f}_{ij;kl}$ is non-negative on $\mathbf{v}_I$.

Conversely, if the restriction of T to i, j, k, and l is the star tree, then there is a vertex v of T for which these are in different parts of $\text{part}(v)$. By choosing a division I at v which keeps i and l together, we get that our functional is negative on $\mathbf{v}_I$. If $T|_{\{i,j,k,l\}}$ is a trivalent tree with i and l as a split, then there is a split of T containing i and l, but not j and k, so the functional is again negative. □

Proposition 3.4. *If T is not a star tree, then C_T is a pointed cone.*

Proof. Let $\mathbf{g}$ be the sum $\sum \mathbf{f}_{ij;kl}$ over all 4-tuples of indices i, j, k, and l such that the restriction of T to these indices is trivalent with i and k as a split. By Proposition 3.3, $\mathbf{g}$ is non-negative on C_T. Let I be a division of T. By replacing I with its complement if necessary, we may assume that the subset of T corresponding to I contains an internal edge e. If we choose j and l not in I and i and k to be on the opposite side of e from j and l, then $\mathbf{f}_{ij;kl}$ is included in the summation forming $\mathbf{g}$ and $\mathbf{f}_{ij;kl} \cdot \mathbf{v}_I = 1$ by (2). Therefore, $\mathbf{g} \cdot \mathbf{v}_I > 0$, and so $\mathbf{g}$ is positive on every ray of C_T. Thus C_T is a pointed cone. □

Subsets $I, J \subset [n]$ are called *incompatible* if all four of the sets $I \cap J$, $I \cap J^c$, $I^c \cap J$, and $I^c \cap J^c$, are non-empty. Conversely, I and J are *compatible* if they are not incompatible. Two key facts about splits that we use are that if $\mathcal{I}$ is a collection of pairwise compatible subsets of $[n]$, then there is a trivalent tree τ for which every element of $\mathcal{I}$ is a split of τ, and, second, that if every split of a tree T is a split of a tree τ, then T is a contraction of τ.

Lemma 3.5. *Let T and T' be two trees in an admissible collection. Suppose I is a division of T that is not a division of T'. Then there exist four indices $i, j, k, l \in [n]$ such that the restrictions $T|_{\{i,j,k,l\}}$ and $T'|_{\{i,j,k,l\}}$ are distinct trivalent trees and $\{i,j,k,l\} \cap I = \{i,j\}$.*

Proof. If I is a split of T that is not a division of T', then we claim that there is a split of T' that is incompatible with I. Indeed, suppose that I was compatible with every split of T'. Then there would be a trivalent tree τ that has I, and every split of T', as splits. But in that case τ would have T' as a contraction, so I would be a division of T'. We can thus find a split J of I that is incompatible with I. Choose $i \in I \cap J$, $j \in I \cap J^c$, $k \in I^c \cap J$, and $l \in I^c \cap J^c$; then $\{i,j,k,l\}$ has the desired properties.

Suppose now that I is a division of T coming from a vertex v of T, and I is not a split. Let $\mathcal{E}$ be the set of edges of T' corresponding to splits that are incompatible with I. Let $\tilde{T}$ be the tree obtained from T by replacing the vertex v with a single edge such that I is the split at that edge. By the previous paragraph applied to $\tilde{T}$ and T' we see that $\mathcal{E}$ is nonempty. Note also that the subgraph of T' consisting of the edges in $\mathcal{E}$ is connected. Indeed, suppose that e and e' are two edges in $\mathcal{E}$

corresponding to splits J and J', and e'' is an edge on the unique path in T' between e and e', corresponding to a split J''. By taking complements if necessary, we may assume that $J \subset J'' \subset J'$. Then $I \cap J'' \neq \emptyset$ and $J'' \setminus I \neq \emptyset$ follow from the same facts for J, and $I \setminus J'' \neq \emptyset$ and $[n] \setminus (I \cup J'') \neq \emptyset$ follow from the same facts for J', so $e'' \in \mathcal{E}$.

Suppose, for the sake of contradiction, that every part H of $\mathrm{part}(v)$ is compatible with every split J of T' corresponding to an edge in $\mathcal{E}$. Let $\tilde{T}'$ be the (possibly trivial) expansion of T' constructed as follows. At each vertex v' of an endpoint of an edge in $\mathcal{E}$, each incident edge not in $\mathcal{E}$ defines a split compatible with I, which is thus either contained in I or disjoint from I. If there are two or more edges incident to v, all of whose splits are contained in I, then we insert an edge at v' separating these edges from the others. Similarly, if there are two or more edges incident to v whose splits are disjoint from I, then we also insert an edge separating these. Note that if v' is incident to only one edge of $\mathcal{E}$, then since the split corresponding to that edge is not compatible with I, the splits corresponding to incident edges not in $\mathcal{E}$ must contain both some contained in I and some disjoint with I, and therefore, we haven't introduced any 2-valent vertices.

We now let T'' be the contraction of $\tilde{T}'$ at all the edges corresponding to edges of $\mathcal{E}$, and let v'' be the vertex resulting from the contraction of these edges. The construction of $\tilde{T}'$ guarantees that every part of $\mathrm{part}(v'')$ will be compatible with I.

Moreover, we claim that $\mathrm{part}(v'')$ coarsens $\mathrm{part}(v)$, and therefore it is admissible. By our assumption that each part H of $\mathrm{part}(v)$ is compatible with the splits corresponding to edges in $\mathcal{E}$, and so for fixed part H, and any two labels $h, h' \in H$, the unique path in T' from the label h to h' must be disjoint from the edges in $\mathcal{E}$. Since I is a division at v, it is a union of parts of $\mathrm{part}(v)$, so H is either contained in I or disjoint from it. Thus our construction of $\tilde{T}'$ ensures that H is contained in a single part of the partition $\mathrm{part}(v'')$ of its contraction T''.

Since $\tilde{T}'$ is an expansion of T', all of its vertices have admissible partitions. All vertices of T'' have the same partition as a vertex of $\tilde{T}'$, except for v'', and $\mathrm{part}(v'')$ was shown to be admissible in the previous paragraph. Therefore, by Lemma 2.6, a contraction of T'' is in $\mathcal{F}$. However, this contraction cannot be T' because any edge in $\mathcal{E}$ gives a split of T' that is not a split of T''. Therefore, we have two distinct contractions of $\tilde{T}'$ contained in the collection $\mathcal{F}$, which contradicts the definition of an admissible collection of trees. We conclude that there exists a part H of $\mathrm{part}(v)$ that is not compatible with some split J of T' corresponding to an edge in $\mathcal{E}$.

Since I is not compatible with J, we have $J \setminus I \neq \emptyset$ and $[n] \setminus (I \cup J) \neq \emptyset$. Since H is not compatible with J, we have $H \cap J \neq \emptyset$, and $H \setminus J \neq \emptyset$. Choose $i \in H \cap J$, $j \in H \setminus J$, $k \in J \setminus I$ and $l \in [n] \setminus (I \cup J)$. Then since H is a part of $\mathrm{part}(v)$, it is a split of T, so the restrictions $T|_{\{i,j,k,l\}}$ and $T'|_{\{i,j,k,l\}}$ are distinct trivalent trees, and $\{i, j, k, l\} \cap I = \{i, j\}$ as required. ☐

Proposition 3.6. *If $\mathcal{F}$ is an admissible collection of trees, then $\{C_T : T \in \mathcal{F}\}$ form the maximal cones of a fan.*

Proof. For any two trees $T, T' \in \mathcal{F}$, we construct a separating linear functional that defines $C_T \cap C_{T'}$ as a face of C_T and $C_{T'}$. Since T and T' do not have a common expansion, the set of splits of T and of T' are not all pairwise compatible. As all

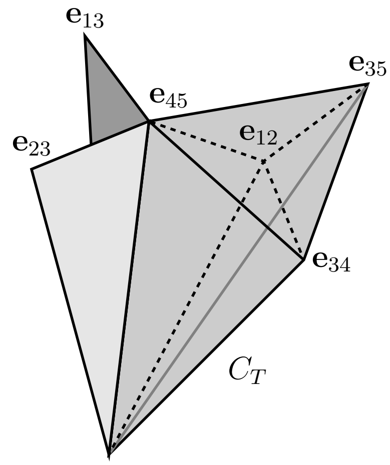

FIGURE 3. Part of the fan $X_{\mathcal{F}}$ if $\mathcal{F}$ is an admissible collection containing a non-trivalent tree T with split $\{1,2\}$, superimposed on top of the fan Δ corresponding to the collection of all trivalent trees, shown with dotted lines. The two 2-dimensional cones corresponding to trivalent trees are common to both fans.

divisions of a tree are compatible with all of its splits, there must be a division of T that is not a division of T', so by Lemma 3.5 there are i, j, k, and l for which the restrictions of T and T' to these indices are both trivalent with splits $\{i,j\}$ an $\{i,k\}$ respectively. Then, $\mathbf{f}_{il;jk} = -\mathbf{f}_{il;kj}$ is non-negative on C_T and non-positive on $C_{T'}$ by Proposition 3.3. We now define $\mathbf{g} \in M$ to be the sum of all $\mathbf{f}_{il;jk}$ for which i,j,k,l has these restriction properties. Then $\mathbf{g}$ is non-negative on C_T and non-positive on $C_{T'}$ which means that the face F of C_T defined by $\mathbf{g}$ satisfies $C_T \cap C_{T'} \subset F$. We claim that this inclusion is an equality.

To see this, suppose that I is a division of T but not T'. Then by Lemma 3.5, there exist indices $i,j \in I$ and $k,l \notin I$ such that T and T' restricted to these four labels are distinct trivalent trees. The restriction $T|_{\{i,j,k,l\}}$ has $\{i,j\}$ as a split, and, without loss of generality, we can assume that $T'|_{\{i,j,k,l\}}$ has $\{i,k\}$ as a split. Therefore, by (2), $\mathbf{f}_{il;jk} \cdot \mathbf{v}_I = 1$, and $\mathbf{g}$ is the sum of $\mathbf{f}_{il;jk}$ and other terms that pair non-negatively with $\mathbf{v}_I$, so $\mathbf{g} \cdot \mathbf{v}_I > 0$. This means that $\mathbf{v}_I$ is not in F, so all the rays of F are contained in $C_T \cap C_{T'}$. As $C_T \cap C_{T'}$ is convex, we have $F = C_T \cap C_{T'}$. □

Definition 3.7. We denote the fan from Proposition 3.6 by $\Sigma_{\mathcal{F}}$. By Proposition 3.4, the cones of $\Sigma_{\mathcal{F}}$ are pointed, and so this fan defines a toric variety, which we denote by $X_{\mathcal{F}}$. For a tree $T \in \mathcal{F}$, we write $U_T \subset X_{\mathcal{F}}$ for the open affine corresponding to the cone C_T.

When $\mathcal{F}$ is the set of all trivalent trees, the fan $\Sigma_{\mathcal{F}}$ is the *space of phylogenetic trees* [MS15, Thm. 4.3.5], which we denote by Δ. This is a pure fan of dimension $n-3$ in $\mathbb{R}^{\binom{n-1}{2}-1}$. In general the fan $\Sigma_{\mathcal{F}}$ is not pure, however.

Example 3.8. We consider now the fans given by the admissible collections with $n = 5$ from Example 2.4. When $\mathcal{F}$ is the collection of all trivalent trees, then $\Sigma_{\mathcal{F}} = \Delta$ has 10 rays and 15 cones of dimension 2. If $\mathcal{F}$ is an arbitrary admissible collection of trees, then each non-trivalent tree T in $\mathcal{F}$ defines a 3-dimensional simplicial cone which replaces three 2-dimensional cones and a ray from Δ. For example, if $\mathcal{F}$ contains the tree T with a single internal edge having split $\{1,2\}$, then

$$C_T = \mathrm{pos}(\mathbf{e}_{12}, \mathbf{e}_{34}, \mathbf{e}_{35}, \mathbf{e}_{45}) = \mathrm{pos}(\mathbf{e}_{34}, \mathbf{e}_{35}, \mathbf{e}_{45}).$$

The second equality come from the relation $\mathbf{e}_{12} = \mathbf{e}_{34} + \mathbf{e}_{35} + \mathbf{e}_{45}$. The cone C_T replaces the cones of Δ corresponding to the 3 trivalent expansions of T together with the ray spanned by $\mathbf{e}_{12}$. This cone and two adjacent ones are shown in Figure 3.

Example 3.9. Consider the admissible collection $\mathcal{F}$ of Example 2.5 corresponding to a numerical equivalence class of F-curves τ, with partition $[n] = A \amalg B \amalg C \amalg D$. Then $\Sigma_{\mathcal{F}}$ has a simplicial $(n-3)$-dimensional cone for each trivalent tree in $\mathcal{F}$, and an $(n-2)$-dimensional cone for each tree T in $\mathcal{F}$ with exactly one non-trivalent vertex whose partition is A, B, C, D. Each of the latter cones is the convex hull of the cones $C_{T'}$ for the three trivalent trees T' that are expansions of T. These cones all intersect in the cone $\mathrm{pos}(\mathbf{e}_A, \mathbf{e}_B, \mathbf{e}_C, \mathbf{e}_D, \mathbf{e}_{AB}, \mathbf{e}_{AC}, \mathbf{e}_{AD})$.

The moduli space $\overline{M}_{0,n}$ can be realized as a subvariety of X_Δ, as we now recall. Let $\mathrm{Gr}(2,n)$ denote the Grassmannian of 2-planes in an n-dimensional vector space, and let $\mathrm{Gr}(2,n)^\circ$ be the open subset of those planes not intersecting any $(n-2)$-dimensional coordinate space, or equivalently of points in $\mathrm{Gr}(2,n)$ for which all Plücker coordinates are nonzero. The n-dimensional torus acts coordinatewise on the ambient vector space, and has an induced action on $\mathrm{Gr}(2,n)$. This action extends to a monomial action on $\mathbb{P}^{\binom{n}{2}-1}$. The quotient of $\mathrm{Gr}(2,n)^\circ$ by this torus action is the moduli space $M_{0,n}$, which embeds into the quotient $\mathbb{T}$ of the torus of $\mathbb{P}^{\binom{n}{2}-1}$. The lattice of characters of $\mathbb{T}$ is the lattice M introduced at the start of the section.

The toric variety X_Δ is a partial compactification of $\mathbb{T}$ such that the closure of $M_{0,n} \subset \mathbb{T}$ in X_Δ equals the moduli space $\overline{M}_{0,n}$ [Tev07, GM10].

Definition 3.10. We write $M_{\mathcal{F}}$ to denote the closure of $M_{0,n}$ inside the toric variety $X_{\mathcal{F}}$ determined by an admissible collection of trees $\mathcal{F}$.

Remark 3.11. In [CHMR16], the authors also construct compactifications of of $M_{0,n}$ by taking the closure inside a toric variety. Specifically, they consider "heavy-light" Hassett weighted moduli spaces $M_{0,\mathbf{a}}$, where each a_i is either very small or equal to 1. Although we will see in Section 7 that Hassett's moduli spaces are tree compactifications, the two constructions use different toric embeddings.

The toric variety in [CHMR16] has a smaller dimension than $X_{\mathcal{F}}$ and its fan is obtained from the space Δ of phylogenetic trees by projecting away some coordinates. Working in this smaller toric variety means that $M_{0,\mathbf{a}}$ can be realized as a tropical compactification in the sense of Tevelev [Tev07]. The ambient toric variety $X_{\mathcal{F}}$ for the tree compactification, by contrast, is birational to the toric variety X_Δ, and the compactification of $M_{0,n}$ is not a tropical compactification.

We now construct morphisms between tree compactifications from morphisms between the underlying toric varieties. Let $\mathcal{F}$ and $\mathcal{F}'$ be admissible collections of trees on n and n' leaves respectively, with $n' \geq n$. We say that $\mathcal{F}'$ is an *expansion* of $\mathcal{F}$ if for every tree T' in $\mathcal{F}'$, the restriction $T'|_{[n]}$ is an expansion of some tree in $\mathcal{F}$. For example, the admissible collection consisting of all trivalent trees on $n' \geq n$ leaves is an expansion of any admissible collection on n leaves.

Proposition 3.12. *Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are admissible collections of trees on n and n' leaves respectively with $n' \geq n$ and $\mathcal{F}'$ expanding $\mathcal{F}$. Then there is a morphism $M_{\mathcal{F}'} \to M_{\mathcal{F}}$, which is birational if $n' = n$.*

Proof. The desired morphism will follow from the construction of a morphism of toric varieties $X_{\mathcal{F}'} \to X_{\mathcal{F}}$. We write $N'_{\mathbb{R}}$ and $N_{\mathbb{R}}$ for the ambient vector spaces of the fans for $X_{\mathcal{F}'}$ and $X_{\mathcal{F}}$, respectively. There is a natural map π from $N'_{\mathbb{R}}$ to $N_{\mathbb{R}}$ that is the quotient of the coordinate projection defined by sending $e_{ij} \in N'_{\mathbb{R}}$ to e_{ij} if $i, j \leq n$ and to $\mathbf{0} \in N_{\mathbb{R}}$ otherwise. For a split $I' \subset [n']$, the image $\pi(\mathbf{v}_{I'})$ equals $\mathbf{v}_I$, where $I = I' \cap [n]$, if $2 \leq |I| \leq n-2$, and is $\mathbf{0}$ otherwise.

To show that π defines a morphism of toric varieties, we need to show that for each $T' \in \mathcal{F}'$, the image $\pi(C_{T'})$ of a cone is contained in a single cone of $X_{\mathcal{F}}$. By above $\pi(C_{T'})$ is the cone spanned by the $\mathbf{v}_I$ as I ranges over the divisions of $T'|_{[n]}$. By assumption, there exists a tree $T \in \mathcal{F}$ that is a contraction of $T'|_{[n]}$, so every division of $T'|_{[n]}$ is also a division of T and thus C_T contains $\pi(C_{T'})$.

Finally, if $n' = n$, then $N'_{\mathbb{R}} = N_{\mathbb{R}}$ and so the map of toric varieties $X_{\mathcal{F}'} \to X_{\mathcal{F}}$ is birational. Since $M_{\mathcal{F}'}$ and $M_{\mathcal{F}}$ meet the dense orbit of their respective toric varieties, the induced morphism $M_{\mathcal{F}'} \to M_{\mathcal{F}}$ is also birational. ☐

We can now prove the existence and basic properties of our compactifications:

Proof of Theorem 1.1. By construction, $M_{\mathcal{F}}$ contains $M_{0,n}$ as a dense open set. Since $M_{\mathcal{F}}$ is the image of the projective variety $\overline{M}_{0,n}$, it must be proper. Finally, the morphism $\phi_{\mathcal{F}}$ exists by Proposition 3.12. It is the restriction of a toric morphism which is an isomorphism on its dense torus, so $\phi_{\mathcal{F}}$ is birational. ☐

4. Stratification of tree compactifications

This section describes a stratification on the compactification $M_{\mathcal{F}}$, induced by the stratification by combinatorial type on $\overline{M}_{0,n}$. Each stratum of $\overline{M}_{0,n}$ maps surjectively onto a single stratum of $M_{\mathcal{F}}$, but a stratum of $M_{\mathcal{F}}$ may be the image of multiple strata in $\overline{M}_{0,n}$. Nonetheless, there is a maximal stratum of $\overline{M}_{0,n}$ mapping to a given stratum of $M_{\mathcal{F}}$, and we can index strata in the latter by a class of trees which we call $\mathcal{F}$-stable.

We first recall the stratification of $\overline{M}_{0,n}$ by combinatorial type. We denote by $\overline{M}_{0,n}^{\tau}$ the locus in $\overline{M}_{0,n}$ consisting of those stable, genus 0 curves with dual graph equal to a given tree τ. The tree τ also labels a cone of the fan Δ, and $\overline{M}_{0,n}^{\tau}$ is the intersection of the torus orbit of X_{Δ} corresponding to this cone with $\overline{M}_{0,n}$. As a first step, we connect the these strata and their trees with the admissible collection $\mathcal{F}$ via the following proposition:

Proposition 4.1. *The locally closed stratum $\overline{M}_{0,n}^{\tau} \subset \overline{M}_{0,n}$ is contracted to a point in $M_{\mathcal{F}}$ if and only if some contraction of τ is in $\mathcal{F}$.*

We begin the proof of Proposition 4.1 with the following lemma, which refines the containment of cones used for the proof of the $n' = n$ case of Proposition 3.12.

Lemma 4.2. *If a tree T' is an expansion of a tree T, then $C_{T'}$ is contained in C_T, but not in any of its proper faces.*

Proof. As in the proof of Proposition 3.12, the containment follows because every division of T' is also a division of T. For the second part, induction on the number of edges of T' contracted to form T shows that it is sufficient to consider the case when T is the contraction of a single edge e of T'. We will show that $C_{T'}$ is not contained in any proper face of C_T by showing that $C_{T'}$ contains a point from the relative interior of C_T.

Let v denote the vertex of T formed by contracting e, and let v', v'' denote the two vertices of e in T'. Let d be the number of parts of $\operatorname{part}(v)$. Since the formula (1) gives $\mathbf{v}_I = 0$ when $2 < |I|$ or $|I| > n-1$ in the following we relax the restriction that $2 \leq |I| \leq n-2$ for a division.

Let $\mathcal{I}$ denote the collection of parts of $\operatorname{part}(v)$. We claim the following identity holds:

$$\sum_{I \text{ a division of } T} \mathbf{v}_I = \sum_{I \text{ a division of } T' \text{ not at } v', v'', I \notin \mathcal{I}} \mathbf{v}_I + 2^{d-3} \sum_{I \in \mathcal{I}} \mathbf{v}_I. \tag{3}$$

Since the left-hand side of (3) is a positive sum over *all* rays of C_T, and the right-hand side is a positive sum over some of the rays of $C_{T'}$, this identity gives a point in $C_{T'}$ in the relative interior of C_T as required.

The difference between the left-hand side and the first sum on the right-hand side is $\sum_{I \text{ a division of } T \text{ at } v} \mathbf{v}_I$. Thus, it suffices to show:

$$\sum_{I \text{ a division of } T \text{ at } v} \mathbf{v}_I = -2^{d-3} \sum_{(i,j) \in \mathcal{D}} \mathbf{e}_{ij} \quad \text{and} \quad \sum_{I \in \mathcal{I}} \mathbf{v}_I = -\sum_{(i,j) \in \mathcal{D}} \mathbf{e}_{ij}, \tag{4}$$

where $\mathcal{D}$ consists of all pairs of indices i and j that are in distinct parts of $\operatorname{part}(v)$. If we fix i and j and suppose that they are in distinct parts K and J, respectively, of $\operatorname{part}(v)$, then $\mathbf{e}_{ij}$ will appear in the definition of $\mathbf{v}_I$ whenever I contains exactly one of K or J. Such divisions I form exactly half of the 2^{d-1} divisions at v, and each contributes $-1/2$ to the sum, so this shows the first equality of (4). For the second identity of (4), the splits in $\mathcal{I}$ correspond to parts of $\operatorname{part}(v)$ and so each $\mathbf{e}_{ij}$ occurs once when choosing the part containing i and once containing j, each with multiplicity $-1/2$. □

We write W_τ for the open affine chart of $\overline{M}_{0,n}$ obtained by intersecting $\overline{M}_{0,n}$ with the open affine chart of X_Δ corresponding to the cone τ of Δ. The open set W_τ is the union of the strata $\overline{M}_{0,n}^{\tau'}$ as τ' ranges over all contractions of τ.

Proof of Proposition 4.1. Consider the closure $V_\tau \subset \overline{M}_{0,n}$ of $\overline{M}_{0,n}^{\tau}$. The image of the stratum $\overline{M}_{0,n}^{\tau}$ under the map $\phi \colon \overline{M}_{0,n} \to M_{\mathcal{F}}$ is a point if and only if the image of V_τ is a point. The closure V_τ is covered by the collection of affine opens W_σ corresponding to all trivalent trees σ that are expansions of τ. Suppose first that there is a tree T in $\mathcal{F}$ contracting τ. By Lemma 4.2 for each such trivalent expansion σ of τ the corresponding cone C_σ is contained in C_T, and so the open affine W_σ is in the preimage of the open affine $M_{\mathcal{F}} \cap U_T$. Thus the image of V_τ is contained in the open affine $M_{\mathcal{F}} \cap U_T$, so must be a point, as V_τ is proper.

Suppose now that $\phi(V_\tau)$ is a point p. Let σ be a trivalent expansion of τ, and let T be the unique tree in $\mathcal{F}$ contracting σ. By Lemma 4.2 the image of the point $\overline{M}_{0,n}^{\sigma}$ is contained in the affine open $M_{\mathcal{F}} \cap U_T$ and not in any other $M_{\mathcal{F}} \cap U_{T'}$ for $T' \in \mathcal{F}$ with $T' \neq T$. Since that image is p for all such σ, the same tree $T \in \mathcal{F}$ must contract all trivalent trees σ refining τ. Therefore, any division of τ is also a division of T, and so T must be a contraction of τ. □

A special case of Proposition 4.1 characterizes which F-curves are contracted by the morphism $\phi_{\mathcal{F}}$. An F-curve corresponds to a tree τ which is trivalent except for a single 4-valent vertex v. By Proposition 4.1, an F-curve is contracted if and only if some contraction of τ is in $\mathcal{F}$. A consequence of the third condition of Definition 2.3 is that whether an F-curve is contracted only depends on the 4-way partition $\operatorname{part}(v)$, and thus if one F-curve is contracted, so are all others numerically equivalent to it.

We now refine Proposition 4.1 by describing in more detail the image in $M_{\mathcal{F}}$ of a stratum $\overline{M}_{0,n}^{\tau}$ when it is not just a point. To do this, we work with *cross-ratios* $x_{ij;kl}$ as coordinate functions on open charts of $M_{\mathcal{F}}$ and $X_{\mathcal{F}}$. Geometrically, the cross-ratios restricted to $M_{0,n}$ are the composition of a forgetful map to $M_{0,4}$ followed by an identification of the latter with $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We follow the convention that this identification comes from taking the point labelled i to ∞, the point labelled j to 0, the point labelled k to 1, and recording the location of the the point labelled l. This means that the cross-ratio of points $(z_i : 1), (z_j : 1), (z_k : 1), (z_l : 1) \in \mathbb{P}^1$ is given by the rational function $(z_i - z_k)(z_j - z_l)/(z_j - z_k)(z_i - z_l)$. On $M_{0,n}$ this translates into the rational function

$$x_{ij;kl} = x_{ik}x_{jl}/x_{il}x_{jk}.$$

Thus $x_{ij;kl}$ is the character of the $(\binom{n}{2} - n)$-dimensional torus corresponding to $\mathbf{f}_{ij;kl} \in M$.

The cross-ratios generate the characters of the torus, and thus the ring of regular functions on $M_{0,n}$, but it is also sufficient to use a proper subset of them.

Lemma 4.3. *For any partition I, I^c into two parts, with $|I|, |I^c| \geq 2$, the set $\{x_{ij;kl} : i, j \in I, k, l \notin I\}$ generates the ring of regular functions on $M_{0,n}$.*

Proof. We need to show that every cross-ratio $x_{ij;kl}$ can be written as a rational function in the given cross-ratios, where the denominator does not vanish on $M_{0,n}$. First suppose that $\{i, j, k, l\}$ has two elements in common with the set I. The relations $x_{ij;kl} = x_{kl;ij} = 1 - x_{ik;jl}$ and $x_{ij;kl} = x_{il;jk}/(x_{il;jk} - 1)$ can then be used to permute the indices so that the first two are in I, and thus write $x_{ij;kl}$ as a rational function of the given cross-ratios. Note that the only denominator introduced has the form $x_{il;jk} - 1$, which is invertible on $M_{0,n}$, as cross-ratios never take the values 0 or 1 on $M_{0,n}$.

Second, suppose that $|\{i, j, k, l\} \cap I|$ has more than two elements. Then, since $x_{ij;kl} = x_{kl;ij}$, we may assume that $i, j \in I$. Choose an index m distinct from i, j, k, l such that $m \notin I$. We then write $x_{ij;kl} = x_{im;kl}x_{mj;kl}$. Since $|\{i, m, k, l\} \cap I| = |\{m, j, k, l\} \cap I| = |\{i, j, k, l\} \cap I| - 1$, after possibly repeating this argument, we have reduced to the first case. By replacing I by I^c, the same argument works when $|\{i, j, k, l\} \cap I| < 2$, which completes the proof. □

Lemma 4.4. *Let T be a tree in an admissible collection $\mathcal{F}$. Then $x_{ij;kl}$ is a regular function on $M_{\mathcal{F}} \cap U_T$ if the restriction $T|_{\{i,j,k,l\}}$ is trivalent with i and l not in the same split.*

Proof. A cross-ratio $x_{ij;kl}$ is a regular function on an affine open $U_T \subseteq X_{\mathcal{F}}$ if and only if $\mathbf{f}_{ij;kl}$ is a non-negative function on C_T, by the characterization of regular functions on a toric variety. The lemma then follows from Proposition 3.3. □

We now describe the image of a stratum $\overline{M}_{0,n}^{\tau}$ in $M_{\mathcal{F}}$. Recall that for any tree τ, the stratum $\overline{M}_{0,n}^{\tau}$ is canonically identified with the product $\prod_v M_{0,\mathrm{link}(v)}$, where v ranges over all non-trivalent vertices of τ and $\mathrm{link}(v)$ refers to the set of edges incident to v. Fix a tree τ and an admissible collection of trees $\mathcal{F}$. We let $\mathcal{A}$ denote the set of vertices v of τ such that $\mathrm{part}(v)$ is an admissible partition of $\mathcal{F}$. Let $\overline{M}_{0,n}^{\tau,\mathcal{A}}$ denote the partial compactification of $\overline{M}_{0,n}^{\tau}$ obtained by taking the union of $\overline{M}_{0,n}^{\tau}$ with $\overline{M}_{0,n}^{\tau'}$ for all expansions τ' of τ at vertices in $\mathcal{A}$. We then have the analogous factorization:

$$\overline{M}_{0,n}^{\tau,\mathcal{A}} \cong \prod_{v\in\mathcal{A}} \overline{M}_{0,\mathrm{link}(v)} \times \prod_{v\notin\mathcal{A}} M_{0,\mathrm{link}(v)} \tag{5}$$

Proposition 4.5. *Fix a tree τ and an admissible collection $\mathcal{F}$ of trees. Let $\mathcal{A}$ be the set of vertices of τ with admissible partitions, as above. Then the restriction of $\phi_{\mathcal{F}}\colon \overline{M}_{0,n} \to M_{\mathcal{F}}$ to $\overline{M}_{0,n}^{\tau,\mathcal{A}}$ is the projection to the factor $\prod_{v\notin\mathcal{A}} M_{0,\mathrm{link}(v)}$ of (5).*

The image $\phi_{\mathcal{F}}(\overline{M}_{0,n}^{\tau})$ is equal to $\phi_{\mathcal{F}}(M_{\mathcal{F}}^{\tau'})$, where τ' is formed by contracting all edges whose refined partition is admissible.

Proof. The fact that the map factors through the projection is vacuous when $\mathcal{A}$ is empty, so for the first part we assume that there is a vertex v in $\mathcal{A}$, with $\mathrm{part}(v)$ is an admissible partition of $\mathcal{F}$. Choose an expansion $\tilde{\tau}$ of τ that does not change $\mathrm{part}(v)$, but is trivalent at the vertices other than v. By Lemma 2.6 there is a contraction of $\tilde{\tau}$ contained in $\mathcal{F}$. By Proposition 4.1, $\overline{M}_{0,n}^{\tilde{\tau}}$ is mapped to a point by $\phi_{\mathcal{F}}$.

Now, let V denote the set of all vertices of τ so that, in the above notation, $\overline{M}_{0,n}^{\tau,V}$ is the closure of $\overline{M}_{0,n}^{\tau}$ in $\overline{M}_{0,n}$. Then we can factor $\overline{M}_{0,n}^{\tau,V}$ as follows:

$$\overline{M}_{0,n}^{\tau,V} = \overline{M}_{0,\mathrm{link}(v)} \times \prod_{v'\neq v} \overline{M}_{0,\mathrm{link}(v')} \tag{6}$$

Note that $\overline{M}_{0,n}^{\tilde{\tau}}$ is contained $\overline{M}_{0,n}^{\tau,V}$, and has the form $\overline{M}_{0,\mathrm{link}(v)} \times x$ for some point x in the second factor of (6). This allows us to apply Mumford's rigidity lemma [Mum08, p. 40]. Recall that this states that if $\psi\colon X \times Y \to Z$ is a morphism of integral varieties, with Y proper, and such that the image of $x \times Y$ is a point, for some point $x \in X$, then the morphism factors through the projection to X. This implies that the restriction of $\phi_{\mathcal{F}}$ to $\overline{M}_{0,n}^{\tau,V}$ factors through the second factor, $\prod_{v'\neq v} \overline{M}_{0,\mathrm{link}(v')}$, of (6). Repeating this argument for all the vertices in $\mathcal{A}$, we get that $\phi_{\mathcal{F}}$ restricted to $\overline{M}_{0,n}^{\tau,V}$ factors through $\prod_{v\notin\mathcal{A}} \overline{M}_{0,\mathrm{link}(v)}$.

It now remains to show, whether $\mathcal{A}$ is empty or not, that the factored map from $\prod_{v\notin\mathcal{A}} \overline{M}_{0,\mathrm{link}(v)}$ to $M_{\mathcal{F}}$ maps the interior $\prod_{v\notin\mathcal{A}} M_{0,\mathrm{link}(v)}$ isomorphically onto its image. Let τ' be a trivalent expansion of τ and let T be the unique contraction of τ' that lies in $\mathcal{F}$. The cone of Δ corresponding to the stratum $\overline{M}_{0,n}^{\tau}$ is contained in $C_{\tau'}$, which is contained in C_T, and so $\phi(M_{0,n}^{\tau})$ is contained in the open $U_T \subset M_{\mathcal{F}}$. The fact that $\phi_{\mathcal{F}}$ factors through the product, and that it is surjective onto its image, means that the coordinate ring of the image injects into the coordinate ring of product $\prod_{v \notin \mathcal{A}} M_{0,\mathrm{link}(v)}$. To complete the proof of the first claim it suffices to show that this injection is also a surjection. We thus want to show that every regular function on $\prod_{v \notin \mathcal{A}} M_{0,\mathrm{link}(v)}$ is the pullback of a regular function on U_T.

To do this, we choose cross-ratios $x_{ij;kl}$ that are regular functions on U_T and that generate the ring of regular functions on $\prod_{v \notin \mathcal{A}} M_{0,\mathrm{link}(v)}$. Pick a non-trivalent vertex $v \notin \mathcal{A}$ of τ, which must exist by Proposition 4.1. In τ' the vertex v is expanded to multiple edges, and, since $\mathrm{part}(v)$ is not admissible, there exists at least one edge e that is not contracted in T. Let I denote the split at e. For any $i, j \in I$ and $j, k \notin I$, the cross-ratio $x_{ij;kl}$ is a regular function on U_T by Lemma 4.4.

Now we consider the restriction of $x_{ij;kl}$ to $\prod_{v' \notin \mathcal{A}} M_{0,\mathrm{link}(v')}$. We claim that this restriction is a non-zero constant times the cross-ratio $x_{e_ie_j;e_ke_l}$ on the factor $M_{0,\mathrm{link}(v)}$, where e_i denotes the edge incident to v in the path from v to i and analogously for e_j, e_k, and e_l. This claim can be seen by considering the modular interpretation of the cross-ratio as the forgetful map to the indices $\{i, j, k, l\}$. If we forget the vertices other than these four, then all the components of a stable curve corresponding to a point of $\overline{M}_{0,n}^{\tau}$ will be contracted except for the one corresponding to the vertex v, on which the marked points i, j, k, and l will end up where the nodes corresponding to e_i, e_j, e_k, and e_l were, so the restriction of $x_{ij;kl}$ is $x_{e_ie_j;e_ke_l}$. These cross-ratios generate the ring of regular functions on $M_{0,\mathrm{link}(v)}$ by Lemma 4.3. Therefore, as v varies over all vertices of τ that are not in $\mathcal{A}$, we get a generating set for all regular functions on $\prod_{v' \notin \mathcal{A}} M_{0,\mathrm{link}(v')}$.

For the second claim, note that a consequence of the first claim is that if τ' is an expansion of τ at vertices v in $\mathcal{A}$, then $\phi(\overline{M}_{0,n}^{\tau}) = \phi(\overline{M}_{0,n}^{\tau'})$. We can thus replace τ with the tree formed by contracting an edge e whenever the vertex formed by the contraction has admissible partition. After repeatedly contracting edges, the refined partition of every edge of τ will be not admissible. $\square$

Definition 4.6. A tree τ is *$\mathcal{F}$-stable* for an admissible collection of trees $\mathcal{F}$ if the refined partition of every edge is not admissible. For an $\mathcal{F}$-stable tree τ, we write $M_{\mathcal{F}}^{\tau}$ for $\phi_{\mathcal{F}}(\overline{M}_{0,n}^{\tau})$.

We finish by showing that the sets $M_{\mathcal{F}}^{\tau}$ stratify $M_{\mathcal{F}}$. As with the strata of $\overline{M}_{0,n}$, these sets are products of lower-dimensional moduli spaces. In particular, by Proposition 4.5,

$$M_{\mathcal{F}}^{\tau} \cong \prod_{v} M_{0,\mathrm{link}(v)},$$

where the product is over all vertices v of τ such that $\mathrm{part}(v)$ is not admissible. To prove the stratification, we need the following variation on Lemma 4.4 to describe the behavior of cross-ratios on neighborhoods of strata $M_{\mathcal{F}}^{\tau}$:

Lemma 4.7. *If τ is an $\mathcal{F}$-stable tree, then $x_{ij;kl}$ is a regular function in a neighborhood of $M_{\mathcal{F}}^{\tau}$ when the restriction $\tau|_{\{i,j,k,l\}}$ is one of the following:*

(1) *a trivalent tree with split $\{i, j\}$, in which case $x_{ij;kl}$ is identically 1 on $M_{\mathcal{F}}^{\tau}$,*

(2) *a trivalent tree with split* $\{i,k\}$, *in which case* $x_{ij;kl}$ *is identically* 0 *on* $M_{\mathcal{F}}^{\tau}$,
(3) *a star tree whose central vertex* v *has non-admissible partition in* τ, *in which case* $x_{ij;kl}$ *ranges over* $\mathbb{A}^1 \setminus \{0,1\}$.

Proof. The stratum $\overline{M}_{0,n}^{\tau}$ of $\overline{M}_{0,n}$ is contained in the affine open W_T for any trivalent expansion T of τ. If τ is in the last case, we choose T to be any expansion such that $T|_{\{i,j,k,l\}}$ does not have $\{i,l\}$ as a split. Let T' be the unique contraction of T in $\mathcal{F}$. If $T'|_{\{i,j,k,l\}}$ is a star tree, then T' contracts all of the edges of T that map to the vertex v of τ, which means that T' contains a vertex whose partition is a refinement of part(v). This contradicts the assumption that part(v) is not admissible, so $T'|_{\{i,j,k,l\}}$ is a trivalent tree without $\{i,l\}$ as a split, and thus $x_{ij;kl}$ is a regular function on $M_{\mathcal{F}} \cap U_{T'}$ by Lemma 4.4.

Now suppose that τ is in one of the first two cases. Let let P be the subtree of τ that becomes the unique internal edge of the restriction $\tau|_{\{i,j,k,l\}}$. In this case, we choose T to be a trivalent expansion of τ such that for each vertex v of P, all of the edges of the expansion of v are added to the path P, so are in the internal edge of $T|_{\{i,j,k,l\}}$. Again, let T' be the unique contraction of T in $\mathcal{F}$. If $T'|_{\{i,j,k,l\}}$ is the star tree, then T' contracts all of the edges in P and all of the expansions of the vertices in P, leaving a vertex whose partition is a refinement of the refined partition of each edge of P. This means that the refined partition of each edge of P is admissible, which contradicts the the assumption that τ is $\mathcal{F}$-stable. Therefore $T'|_{\{i,j,k,l\}}$ is trivalent, and so $x_{ij;kl}$ is a regular function, again by Lemma 4.4.

To prove the statements about the range of $x_{ij;kl}$, it suffices to look at the cross-ratio on the stratum $\overline{M}_{0,n}^{\tau}$, because this projects surjectively onto $M_{\mathcal{F}}$. By definition, $\overline{M}_{0,n}^{\tau}$ consists of points corresponding to stable curves with dual graph equal to τ. The cross-ratio is a coordinate for the forgetful map defined by the 4 indices i, j, k, and l. This forgetful map sends a stable curve with dual graph τ to a curve whose dual graph is the restriction $\tau|_{\{i,j,k,l\}}$. In the first two cases from the lemma statement, this restriction is a trivalent tree, corresponding to a single point on $\overline{M}_{0,4}$, and verifying the particular value of the cross-ratio is a computation. In the third case, $\tau|_{\{i,j,k,l\}}$ is a star tree, so the image of the forgetful map is $M_{0,4}$, which is isomorphic to $\mathbb{A}^1 \setminus \{0,1\}$ via the cross-ratio, as claimed. □

Proposition 4.8. *The compactification* $M_{\mathcal{F}}$ *is stratified by the locally closed sets* $M_{\mathcal{F}}^{\tau}$ *as* τ *ranges over all* $\mathcal{F}$*-stable trees. The closure of* $M_{\mathcal{F}}^{\tau}$ *is the union of all strata* $M_{\mathcal{F}}^{\tau'}$ *for which there exists an expansion* τ'' *of* τ *that is also an expansion of* τ' *at vertices with admissible partitions.*

Proof. Since $\overline{M}_{0,n}$ is stratified by the $\overline{M}_{0,n}^{\tau}$, $M_{\mathcal{F}}$ is the union of $\phi_{\mathcal{F}}(\overline{M}_{0,n}^{\tau})$ as τ ranges over all trees τ. By Proposition 4.5 we can contract all edges of τ whose refined partitions are admissible without changing the image $\phi_{\mathcal{F}}(\overline{M}_{0,n}^{\tau})$ and thus, $\phi_{\mathcal{F}}(\overline{M}_{0,n}^{\tau})$ equals $M_{\mathcal{F}}^{\tau'}$ for τ' the maximal such contraction and these sets cover $M_{\mathcal{F}}$.

Let τ and τ' be distinct $\mathcal{F}$-stable trees. We first show that $M_{\mathcal{F}}^{\tau}$ and $M_{\mathcal{F}}^{\tau'}$ are disjoint. Suppose that there exists a split I of τ which is incompatible with some split J of τ'. Then we can choose i, j, k, and l from $I \cap J$, $I \cap J^c$, $I^c \cap J$, and $I^c \cap J^c$ so that $x_{ij;kl}$ is defined on an open set containing both $M_{\mathcal{F}}^{\tau}$ and $M_{\mathcal{F}}^{\tau'}$ by Lemma 4.7.

Moreover, $x_{ij;kl}$ takes distinct values on these two sets, since the restrictions $\tau|_{\{i,j,k,l\}}$ and $\tau'|_{\{i,j,k,l\}}$ are distinct trivalent trees.

On the other hand, suppose that every split of τ is compatible with every split of τ'. Then, the union of these splits defines a tree T which is an expansion of both τ and τ'. Suppose further that this expansion only expands vertices of τ and τ' with admissible partitions. In that case, since τ and τ' are $\mathcal{F}$-stable, they must be both be the contraction of all edges of T whose refined partitions are admissible. This contradicts our assumption that τ and τ' are distinct.

Therefore, without loss of generality, we may assume that T expands a vertex v of τ such that $\operatorname{part}(v)$ is not admissible. Choose an edge of T expanding v, with corresponding split I, and pick indices $i, k \in I$, $j, l \in I^c$. Since I is not a split of τ, it must be a split of τ'. By Lemma 4.7, $x_{ij;kl} = 0$ on $M_{\mathcal{F}}^{\tau'}$, but it is non-zero on $M_{\mathcal{F}}^{\tau}$ since $\tau|_{\{i,j,k,l\}}$ is a star tree. This again shows that $M_{\mathcal{F}}^{\tau}$ and $M_{\mathcal{F}}^{\tau'}$ are disjoint.

Finally, we need to prove the claim about the closure of $M_{\mathcal{F}}^{\tau}$. Since $M_{\mathcal{F}}^{\tau} = \phi_{\mathcal{F}}(\overline{M}_{0,n}^{\tau})$ and the closure of $\overline{M}_{0,n}^{\tau}$ is proper and is the union of all strata indexed by expansions of τ, the closure of $M_{\mathcal{F}}^{\tau}$ is the union of $\phi_{\mathcal{F}}(\overline{M}_{0,n}^{\tau''})$ as τ'' ranges over expansions of τ. If we contract the edges of τ'' whose refined partitions are admissible, then $\phi_{\mathcal{F}}(\overline{M}_{0,n}^{\tau''}) = M_{\mathcal{F}}^{\tau'}$ for some $\tau' \in \mathcal{F}$, and so we obtain the description from the proposition statement. □

5. Smyth modular compactifications

In [Smy13] Smyth gives a combinatorial construction, indexed by an "extremal assignment", of a modular compactification of $\overline{M}_{g,n}$. In this section we show that in genus zero these are, up to normalization, examples of tree compactifications. We recall our convention from Definition 2.1 that a tree always has internal vertices of degree at least 3 and its leaves are labeled by $[n]$. A genus zero extremal assignment assigns to each tree T a subset $Z(T)$ of the internal vertices of T such that:

(1) For any tree T, there is at least one internal vertex not in $Z(T)$.
(2) For any map of trees $\phi : T' \to T$, we have $v \in Z(T)$ if and only if $\phi^{-1}(v) \subseteq Z(T')$.

Smyth's description for general g has the extra requirement that for a graph G of genus g, $Z(G)$ is invariant under $\operatorname{Aut}(G)$. This condition is vacuous in genus 0, as trees with labelled leaves have no automorphisms.

Trees with labelled leaves are the set of dual graphs of stable genus 0 curves with n marked points. For a curve C with dual graph T, the set $Z(T)$ defines a distinguished collection of irreducible components of the curve. A reduced, marked, genus 0 curve $(C, p_1, \dots, p_n)$ is Z-stable if there is $(C', p'_1, \dots, p'_n)$ and a surjective morphism $\pi \colon C' \to C$ with $\pi(p'_i) = p_i$ and connected fibers such that π contracts a connected component Y of the distinguished collection of components of C' to a (possibly singular) point of C with multiplicity $|Y \cap Y^c|$. In [Smy13, Theorem 1.21] Smyth shows that the moduli space $\mathcal{M}_{0,n}(Z)$ of Z-stable curves is an algebraic space.

We first show that all such compactifications are, up to normalization, tree compactifications. From an extremal assignment Z, we construct an admissible collection of trees $\mathcal{F}_Z$ with n labels and an admissible collection $\widetilde{\mathcal{F}}_Z$ of trees on $n+1$ labels. We

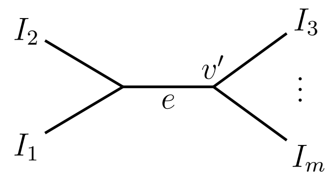

FIGURE 4. The tree T' used in the proof of Proposition 5.2 to show that the partitions of Z are closed under coarsening.

will show that $M_{\mathcal{F}_Z}$ is the normalization of $\mathcal{M}_{0,n}(Z)$, and $M_{\widetilde{\mathcal{F}}_Z}$ is the universal family of this moduli space.

Definition 5.1. Let Z be an extremal assignment. Set $\mathcal{F}_Z$ to be the set of all trees T with n leaves such that no two vertices in $Z(T)$ are adjacent and such that all non-trivalent internal vertices of T are in $Z(T)$. Set $\widetilde{\mathcal{F}}_Z$ to be the set of all of trees $\widetilde{T}$ with $n+1$ leaves such that the restriction $T = \widetilde{T}|_{[n]}$ is in $\mathcal{F}_Z$ and $\widetilde{T}$ is the result of attaching $n+1$ either to a vertex of $Z(T)$ or to an edge that is not incident to any vertex of $Z(T)$.

Proposition 5.2. *Both $\mathcal{F}_Z$ and $\widetilde{\mathcal{F}}_Z$ are admissible collections of trees.*

Proof. We prove both claims by appeal to Proposition 2.10. Let $\mathcal{P}$ be the set of all partitions that either have at most 3 parts or are of the form $\operatorname{part}(v)$ for a vertex v in $Z(T)$ for some tree T. Note that if T' is any other tree with a vertex v' such that $\operatorname{part}(v) = \operatorname{part}(v')$, then T and T' both have a common contraction formed by contracting all internal edges not incident to v or v'. By the assumptions on an extremal assignment, v' is also in $Z(T')$. In other words, whether a vertex v of a tree T is in $Z(T)$ only depends on the partition $\operatorname{part}(v)$. We will refer to such partitions as the *partitions of Z*.

Now we check the conditions in Definition 2.7. First, any partition into at most 3 parts is in $\mathcal{P}$ by assumption. Second, if T is the star tree, then the definition of an extremal assignment makes the assumption that the central vertex is not in $Z(T)$ and therefore $\{1\}, \ldots, \{n\}$ is not in $\mathcal{P}$.

For the third condition, we claim that if σ' is a coarsening of σ with at least 3 parts, and σ is a partition of Z, then σ' is also a partition of Z. Since partitions σ' with at most 3 parts are in $\mathcal{P}$ automatically, this claim will suffice to prove that $\mathcal{P}$ is closed under coarsening. It is enough to prove the claim when σ' is formed by merging the first two parts of σ, whose parts we denote by $I_1, \ldots, I_m$. In that case, let T' be a tree as in Figure 4, where the edge e has refined partition σ and let T be the contraction of the edge e to form a vertex v. Since σ is assumed to be a partition of Z, we have $v \in Z(T)$, and so its two preimages are also in $Z(T')$. Therefore, $\sigma' = \operatorname{part}(v')$ is a partition of Z, which proves the claim.

For the fourth condition, suppose that $\pi = I_1, \ldots, I_l$ is a partition and r and s are positive integers such that $r + s < l$ and the two partitions formed by merging the first r and the last s parts of π, respectively, are both in $\mathcal{P}$. Consider the tree T as in Figure 2, where the refined partition of e is π. Let T' and T'' be the trees formed by contracting either of the edges e and e', respectively. The vertices formed by the contracting e or e' have partitions in $\mathcal{P}$, so are in $Z(T')$ and $Z(T'')$, respectively.

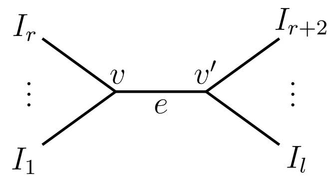

FIGURE 5. The tree T used in the proof of Proposition 5.2 for checking the fourth axiom of admissible partitions for the set $\tilde{\mathcal{P}}$.

Therefore, the three vertices of e and e' of T are also similarly in $Z(T)$ and so the vertex formed by contracting both e and e' is in $Z(T''')$. Since the partition of this vertex is $I_1, \ldots, I_l$, this shows that π is in $\mathcal{P}$, as desired, and so $\mathcal{F}_Z$ is an admissible collection of trees.

Let $\tilde{\mathcal{P}}$ be the collection of partitions π of $[n+1]$ such that either π has at most three parts, or $n+1$ is not a part of π and the result of removing $n+1$ from π lies in $\mathcal{P}$, or $n+1$ is a part of π and the result of removing $n+1$ from π is a partition of Z. We next show that this is an admissible collection of partitions, and $\tilde{\mathcal{F}}_Z$ is the corresponding admissible collection of trees.

We first show that this is indeed an admissible collection of partitions. Again, the first axiom follows from the construction and the second follows from the fact that the central vertex of a star tree T is not in $Z(T)$.

If $\sigma \in \tilde{\mathcal{P}}$, then the restriction to $[n]$ of a coarsening σ' of σ is a coarsening of the restriction of σ', so lies in $\tilde{\mathcal{P}}$ because $\mathcal{P}$ is an admissible collection of partitions. This shows the third axiom.

For, the third axiom, we take a coarsening σ' of $\sigma \in \tilde{\mathcal{P}}$. We may assume that σ' has at least 4 parts. The result of removing $n+1$ from σ' is a coarsening of the result of removing $n+1$ from σ. This suffices to show that $\sigma' \in \tilde{\mathcal{P}}$ if $n+1$ is not a part of σ', as the result of removing $n+1$ from $\sigma \in \tilde{\mathcal{P}}$ lies in $\mathcal{P}$. If $n+1$ is a part of σ', then it is also a part of σ, and so the result of removing $n+1$ from σ is a partition of Z. The same is true for any coarsening, so again $\sigma' \in \tilde{\mathcal{P}}$.

For the fourth axiom, we suppose that $\pi = I_1, \ldots, I_l$ is a partition of $[n+1]$ and r and s are positive integers with $r + s \leq l - 1$ such that merging the first r or last s parts of π yields partitions in $\tilde{\mathcal{P}}$. If r or s equals 1, then the condition is trivial, and so we can assume that $r, s \geq 2$, which means that the partitions after merging each have at least 4 parts. If $n+1$ is not a part of the merged partition then this follows from the same axiom for $\mathcal{P}$, so we may assume that $n+1$ is one of $I_{r+1}, \dot{;} I_{l-s}$. If $r + s < l - 1$, then the result of removing $n+1$ from the original partition is again a partition satisfying the conditions of this axiom for $\mathcal{P}$, and we get that the result of removing $n+1$ from the merged partition is again a partition of Z by the argument given for $\mathcal{P}$. We may thus reduce to the case that $r + s = l - 1$.

We now assume that $r + s = l - 1$, with $I_{l-s} = I_{r+1} = \{n+1\}$. Consider the tree T shown in Figure 5. If $I_1, \ldots, I_r, \{n+1\}, I_{r+2} \cup \cdots \cup I_l$ is in $\tilde{\mathcal{P}}$, then the vertex v of T must be in $Z(T)$, and similarly if $I_1 \cup \cdots \cup I_r, \{n+1\}, I_{r+2}, \ldots, I_l$ is in $\tilde{\mathcal{P}}$, then v' is in $Z(T)$. Contracting the edge e of T yields a tree T' with a vertex in $Z(T')$, showing that $I_1, \ldots, I_l$ is in $\tilde{\mathcal{P}}$. $\square$

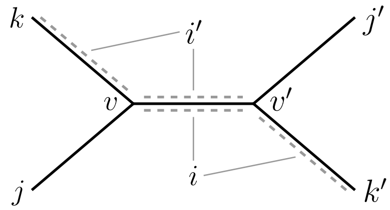

FIGURE 6. Arrangement of the leaves i, j, k, i', j', and k' on the tree T considered in the proof of Lemma 5.3. The leaves i and i' can be placed anywhere in the regions indicated by dashed lines, not including the vertices v or v', which are assumed to be trivalent.

We will show that the map $M_{\widetilde{\mathcal{F}}_Z} \to M_{\mathcal{F}_Z}$ is a flat family satisfying Smyth's moduli problem for the extremal assignment Z. We first prove the following step in that direction:

Lemma 5.3. *The map $M_{\widetilde{\mathcal{F}}_Z} \to M_{\mathcal{F}_Z}$ is a flat family with reduced fibers.*

Proof. The statement is local on $M_{\mathcal{F}_Z}$, so it suffices to work with an affine chart $U_T \cap M_{\mathcal{F}_Z}$ corresponding to a single tree $T \in \mathcal{F}_Z$. We write $W \subset M_{\widetilde{F}_Z}$ for the preimage of $U_T \cap M_{F_Z}$. Let $\mathcal{S}$ denote the set of unordered triples $\{i,j,k\} \subset \{1,\dots,n\}$ such that the unique internal vertex of the restriction of T to $\{i,j,k\}$ is not in $Z(T)$. For $\{i,j,k\} \in \mathcal{S}$, the forgetful morphism $M_{0,n+1} \to M_{0,\{i,j,k,n+1\}}$ extends to a morphism $W \to \overline{M}_{0,\{i,j,k,n+1\}}$. Indeed, coordinates on $M_{0,\{i,j,k,n+1\}}$ can be given by the cross-ratio $x_{ij;k(n+1)}$ or a cross-ratio given by a permutation of these indices. Then, Lemma 4.4 shows that one of these permutations is a regular function on $U_{\widetilde{T}} \cap M_{\widetilde{\mathcal{F}}_Z}$ for any $\widetilde{T}$ having T as a restriction. Let $\psi\colon W \to (U_T \cap M_{\mathcal{F}_Z}) \times \prod_{ijk\in\mathcal{S}} \overline{M}_{0,\{i,j,k,n+1\}}$ be the product of the projection and these forgetful morphisms. We will show that ψ is a closed immersion, which means that the fibers of $W \to U_T \cap M_{\mathcal{F}_Z}$ are closed subschemes of $\prod_{ijk\in\mathcal{S}} \overline{M}_{0,\{i,j,k,n+1\}}$.

We now want to show that the fibers of $W \to U_T \cap M_{\mathcal{F}_Z}$ are reduced subschemes of the product $\prod_{ijk\in\mathcal{S}} \overline{M}_{0,\{i,j,k,n+1\}} \cong (\mathbb{P}^1)^S$. We do that by finding explicit equations for W as a subscheme of $(U_T \cap M_{\mathcal{F}_Z}) \times \prod_{\{ijk\in\mathcal{S}\}} \overline{M}_{0,\{i,j,k,n+1\}}$, which give a Gröbner basis on each fiber, as a subscheme of $\prod_{\{ijk\in\mathcal{S}\}} \overline{M}_{0,\{i,j,k,n+1\}}$. This Gröbner basis will show that the fibers are reduced and all have the same Hilbert function, from which we can conclude that $W \to U_T \cap M_{\mathcal{F}_Z}$ is flat.

Let $\{i,j,k\}$ and $\{i',j',k'\}$ be two triples in $\mathcal{S}$ and we will given an equation on $(U_T \cap M_{\mathcal{F}_Z}) \times \overline{M}_{0,\{i,j,k,n+1\}} \times \overline{M}_{0,\{i',j',k',n+1\}}$ which pulls back to an equation on $U \times (\mathbb{P}^1)^S$ which vanishes on $\psi(W)$. We let v and v' be the unique internal vertices of the restriction of T to these triples $\{i,j,k\}$ and $\{i',j',k'\}$, respectively. By the definition of $\mathcal{F}_Z$, both v and v' must be trivalent vertices of T. If v and v' are distinct, we can assume that the path from v to v' shares edges with the path from v to i and also with the path from v' to i'. Moreover, we can assume that the path from v to j and from v to i' don't share an edge, and similarly with the paths from v' to j' and i. Thus, T restricted to these labels is as depicted in Figure 6.

We claim that the following equation on $(U_T \cap M_{\mathcal{F}_Z}) \times \overline{M}_{0,\{i,j,k,n+1\}} \times \overline{M}_{0,\{i',j',k',n+1\}}$ pulls back to an element of the ideal of the image of ψ:

$$u_{j'i'k'}u_{jik} - x_{j'i';k'j}w_{j'i'k'}u_{jik} - x_{ji;kj'}u_{j'i'k'}w_{jik} + x_{j'i';k'j}x_{ji;ki'}w_{j'i'k'}w_{jik} \tag{7}$$

Here we give the equation using u_{jik} and w_{jik} as homogeneous coordinates on $\overline{M}_{0,\{i,j,k,n+1\}}$ such that the cross-ratio is equal is equal to the rational function u_{jik}/w_{jik}. One can check that (7) vanishes on the image of $M_{0,n+1}$ by reweriting it in terms of the x variables and then expanding those using their definitions as cross-ratios. We also need to show that each of the variables $x_{j'i';k'j}$, $x_{ji;kj'}$, and $x_{ji;ki'}$ are regular functions on $M_{\mathcal{F}_Z} \cap C_T$. For the first two variables, this is because the corresponding restrictions $T|_{\{j',i',k',j\}}$ and $T|_{\{j,i,k,j'\}}$ have splits $\{j',k'\}$ and $\{j,k\}$, respectively, and the claim then follows from Lemma 4.4. For $x_{ji;ki'}$, the restriction $T|_{\{j,i,k,i'\}}$ to these four labels can have either the splits $\{j,k\}$ or $\{j,i\}$, but Lemma 4.4 proves the claim in either case.

Therefore, if we consider a fixed point of $U_T \cap M_{\mathcal{F}_Z}$, then its fiber in W can be considered as a closed subscheme of $\prod_{\{i,j,k\}\in\mathcal{S}} \overline{M}_{0,\{i,j,k,n+1\}}$ whose defining ideal I contains a bilinear equation for every pair of factors, which is obtained by specializing the x variables in (7). In any term order for which $u_{jik} > w_{jik}$ for all triples $\{i,j,k\}$ in $\mathcal{S}$, the initial ideal of I contains $u_{j'i'k'}u_{jik}$ for all pairs of a triple $\{i,j,k\}$ and a triple $\{i',j',k'\}$ in $\mathcal{S}$. These monomials define a reduced union of coordinate lines in the product $(\mathbb{P}^1)^S$ with multidegree $(1,1,\dots,1)$. Any larger ideal would have a smaller multidegree, and thus to show that these monomials $u_{j'i'k'}u_{jik}$ generate the entirety of the initial ideal of I, it is sufficient to show that I has multidegree $(1,1,\dots,1)$. However, over a generic point of $U_T \cap M_{\mathcal{F}_Z}$, the fiber is a smooth $\mathbb{P}^1$, which maps isomorphically onto each factor $\overline{M}_{0,\{i,j,k,n+1\}}$, and therefore embedding of the generic fiber under ψ has multidegree $(1,1,\dots,1)$. By the upper semicontinuity of multidegrees in projective families, I must also have multidegree $(1,1,\dots,1)$, and so is generated by bilinear equations, which form a Gröbner basis.

Since the ideal I has the same square-free initial ideal for any point in $U_T \cap M_{\mathcal{F}_Z}$, I is radical and has constant Hilbert function. Therefore, the morphism $W \to U_T \cap M_{\mathcal{F}}$ is flat and its fibers are reduced, which completes the proof. ☐

The second ingredient we need in order to satisfy Smyth's moduli problem is to construct the sections of the map $M_{\widetilde{F}_Z} \to M_{F_Z}$ corresponding to the marked points. The images of these sections are closures of strata as in Section 4, and the morphism is given by the following lemma.

Lemma 5.4. *Let τ_i denote the tree having a single internal edge with split $\{i, n+1\}$. The closure D_i of the stratum $M^{\tau_i}_{\widetilde{F}_Z}$ is a section of the map $M_{\widetilde{F}_z} \to M_{F_Z}$.*

Proof. We exhibit a section morphism $M_{\mathcal{F}_Z} \to M_{\widetilde{F}_Z}$ whose image is the divisor D_i using the restriction of a toric morphism. We first describe the closed toric subvariety of $X_{\widetilde{F}_Z}$ corresponding to the ray $\mathbf{v}_{\{i,n+1\}}$. The fan of this toric variety lives in the quotient vector space $N_{\mathbb{R}}/\langle \mathbf{v}_{\{i,n+1\}}\rangle$, where $\langle \mathbf{v}_{\{i,n+1\}}\rangle$ is the span of $\mathbf{v}_{\{i,n+1\}}$, and has a cone $C_{\widetilde{T}}/\langle \mathbf{v}_{\{i,n+1\}}\rangle$ for each tree $\widetilde{T} \in \widetilde{F}_Z$ for which $\{i, n+1\}$ is a division. Such trees are in bijection with the trees T of $\mathcal{F}$ by attaching the label $n+1$ either to the internal vertex adjacent to i or to the leaf edge incident to i, depending on whether the vertex is or is not in $Z(T)$, respectively.

If we instead look at the fan Δ_{n+1} of phylogenetic trees with $n+1$ labels, then the quotients of the cones of Δ_{n+1} that contain the ray $\mathbf{v}_{\{i,n+1\}}$ form the fan Δ_n, embedded in a linear subspace of dimension $\binom{n-1}{2}-1$ in $N_{\mathbb{R}}/\mathbf{v}_{\{i,n+1\}}$. This is a combinatorial analogue of the factorization of boundary strata of $\overline{M}_{0,n}$ given in the text after Lemma 4.4. In particular, the image of the rays $\mathbf{v}_I$ corresponding to splits I compatible with $\{i, n+1\}$ are arranged as the rays of Δ_n. We claim that if we construct the fan $X_{\mathcal{F}}$ using these rays, then each cone is contained in a single cone of the link of $\mathbf{v}_{\{i,n+1\}}$ in $X_{\widetilde{\mathcal{F}}}$. This is the combinatorial input necessary for the claimed toric morphism.

To see the claim, let $\widetilde{T}$ be a tree in $\widetilde{F}_Z$ and let $T = \widetilde{T}|_{[n]} \in \mathcal{F}_Z$ be its restriction. If $\widetilde{T}$ is formed by attaching $n+1$ to a leaf edge incident to i in T, then $C_{\widetilde{T}}$ is the cone generated by the vectors $\mathbf{v}_I$ where I is compatible with $\{i, n+1\}$, and its quotient $C_{\widetilde{T}}/\mathbf{v}_{\{i,n+1\}}$ is naturally identified with C_T by the previous paragraph. On the other hand, if $\widetilde{T}$ is formed by attaching $n+1$ to a vertex of T, then let $\widetilde{T}'$ be the expansion of $\widetilde{T}$ at that vertex which has $\{i, n+1\}$ as a split. Then $C_{\widetilde{T}'} \subset C_{\widetilde{T}}$, and $C_{\widetilde{T}'}/\mathbf{v}_{\{i,n+1\}}$ can be identified with C_T, as before, and so we have the desired inclusion, which gives us the necessary section morphism. □

Proposition 5.5. *The family $M_{\widetilde{\mathcal{F}}_Z} \to M_{\mathcal{F}_Z}$ satisfies Smyth's moduli problem for the extremal assignment Z.*

Proof. By Lemma 5.3, the map $M_{\widetilde{\mathcal{F}}_Z} \to M_{\mathcal{F}_Z}$ is flat and its fibers are reduced, and by Lemma 5.4, we have the necessary section morphism, so we just need to check that the fibers have the right topology, using Proposition 4.8.

We fix a tree τ on $[n]$ corresponding to a stratum $M^{\tau}_{\mathcal{F}_Z} \subset M_{\mathcal{F}_Z}$. The preimage of this stratum in $M_{\widetilde{\mathcal{F}}_Z}$ is the union of strata $M^{\widetilde{\tau}}_{\widetilde{F}}$ such that $\widetilde{\tau}$ is a tree on $[n+1]$, satisfying the conditions of Proposition 4.8, and agreeing with τ when restricted to $[n]$. We claim that if $\widetilde{\tau}$ is obtained from τ by attaching the label $n+1$ to either a vertex or edge of τ, then $\widetilde{\tau}$ satisfies the conditions of Proposition 4.8 if and only if $n+1$ is not attached to an edge e of τ with an endpoint v which is contained in $Z(\tau)$.

To prove the "only if" direction of this claim, we suppose that in $\widetilde{\tau}$, $n+1$ is attached to an edge e with an endpoint $v \in Z(\tau)$. Let $\widetilde{e}$ be the subdivision of e containing v, and then the partition of $\widetilde{e}$ consists of $\{n+1\}$ together with the parts of $\operatorname{part}(v)$. On the other hand, we can find an expansion T of τ such that $T \in \mathcal{F}_Z$ and there is a vertex $v' \in T$ with $\operatorname{part}(v') = \operatorname{part}(v)$ and $v' \in Z(T)$. Thus, we can attach $n+1$ to the vertex v' to get a tree $\widetilde{T} \in \widetilde{\mathcal{F}}_Z$. The partition of v' in this tree is equal to the partition of v' in T together with $\{n+1\}$, so $\widetilde{\tau}$ does not label a stratum of $M_{\widetilde{F}}$ in the sense of Proposition 4.8.

Conversely, suppose that $\widetilde{\tau}$ is obtained from τ by attaching $n+1$ to either a vertex or an edge whose endpoints are both not in $Z(\tau)$. If we attach $n+1$ to a vertex, then any edge partition of $\widetilde{\tau}$, restricted to $[n]$, is an edge partition of τ. Any such edge partition is not admissible, so $\widetilde{\tau}$ is a valid stratum tree. On the other hand, if we attach e to an edge with endpoints $v, w \notin Z(\tau)$, then the edge partitions for the subdivisions of e, restricted to $[n]$, are $\operatorname{part}(v)$ and $\operatorname{part}(w)$, which are not admissible.

These strata are 1-dimensional fibrations over $M^{\tau}_{\mathcal{F}_Z}$ when $n+1$ is attached to a vertex $v \notin Z(\tau)$ and 0-dimensional otherwise, which shows that the fibers have the appropriate tree structure for Smyth's moduli problem. □

Proof of Theorem 1.2. Let Z be an extremal assignment. By Proposition 5.5, the family $M_{\tilde{\mathcal{F}}_Z} \to M_{\mathcal{F}_Z}$ satisfies the moduli problem associated to the extremal assignment Z, so there is a morphism $M_{\mathcal{F}_Z} \to M_{0,n}(Z)$. We show that this map is bijective by looking at a single stratum $M^{\tau}_{\mathcal{F}_Z}$ for an $\mathcal{F}_Z$-stable curve τ. A curve C corresponding to a point in $\overline{M}^{\tau}_{0,n}$ has dual graph τ. The stabilization of C corresponding to a point in $M_{0,n}(Z)$ contracts all the components corresponding to vertices in $Z(\tau)$. Thus, the image of $\overline{M}^{\tau}_{0,n}$ in $M_{0,n}(Z)$ is isomorphic to the product $\prod_{v \in V(\tau)\setminus Z(\tau)} M_{0,\mathrm{link}(v)}$. Since $Z(\tau)$ is the same as the set of admissible vertices of τ, this image is isomorphic to $M^{\tau}_{\mathcal{F}_Z}$ by Proposition 4.5, showing that $M^{\tau}_{\mathcal{F}_Z}$ maps bijectively onto its image in $M_{0,n}(Z)$.

Finally, we need to show that the image of a different stratum is disjoint from the image of $M^{\tau}_{\mathcal{F}_Z}$ in $\overline{M}_{0,n}(Z)$. The curves that have the same stabilization as C either have the same dual graph, or their dual graph is an expansion of τ at an admissible vertex. Such expansions are not $\mathcal{F}_Z$-stable, so no other stratum map to the point corresponding to C in $\overline{M}_{0,n}(Z)$.

Therefore, $M_{\mathcal{F}_Z} \to M_{0,n}(Z)$ is bijective. Since it is also proper, it is finite, and thus factors through the normalization of $M_{0,n}(Z)$, which coincides with the normalization of $M_{\mathcal{F}_Z}$, as claimed. □

6. Giansiracusa-Jensen-Moon's GIT compactifications

In [GJM13] Giansiracusa, Jensen, and Moon studied compactifications of $M_{0,n}$ coming from GIT quotients. For generic linearizations, these compactifications are modular compactifications, as in the last section, and thus, up to normalization, tree compactifications. However, there is also a natural way to associate tree compactifications to the compactifications coming from non-generic linearizations, such that there exists a morphism from the normalization of the associated tree compactification to the GIT quotient.

Let $U_{d,n} = \{(X, p_1, \dots, p_n) \in C_d \times (\mathbb{P}^d)^n : p_i \in X \text{ for all } i\}$. There is an action of $\mathrm{SL}(d+1)$ on $U_{d,n}$, given by change of coordinates on each factor. Then, for suitable linearizations, the GIT quotient $U_{d,n} /\!\!/ \mathrm{SL}(d+1)$ is a compactification of $M_{0,n}$ [GJM13, Thm. 1.1].

The construction of the GIT quotient depends on the degree d as well as linearization parameters $\gamma \in \mathbb{Q}$, and $\mathbf{c} \in \mathbb{Q}^n$, which satisfy:

$$d \geq 1 \quad 0 < \gamma, c_i < 1 \quad (d-1)\gamma + c_{[n]} = d+1, \tag{8}$$

where for any subset $I \subset [n]$, we define

$$c_I = \sum_{i \in I} c_i$$

Motivated by their stability analysis, in [GJM13, Sec. 3.1], the authors define the function:

$$\phi(I, \gamma, \mathbf{c}) = \frac{c_I - 1}{1 - \gamma}$$

We define the following function, which differs from σ in [GJM13, Sec. 3.1] only along the walls of the GIT parameters:

$$\sigma'(I) = \begin{cases} 0 & \text{if } c_I < 1 \\ \lfloor \phi(I, \gamma, \mathbf{c}) \rfloor + 1 & \text{if } 1 \leq c_I < c_{[n]} - 1 \\ d & \text{if } c_I \geq c_{[n]} - 1 \end{cases} \tag{9}$$

By the proof of Lemma 3.1 in [GJM13], $\phi(I, \gamma, \mathbf{c}) \leq d - 1$, and so $\sigma'(I) \leq d$.

If $I_1, \dots, I_m$ is a partition of $[n]$, we define $\sigma'(I_1, \dots, I_m)$ to be $\sum_{j=1}^m \sigma'(I_j)$. The following lemma is similar to [GJM13, Lemma 3.2].

Lemma 6.1. *Let $I_1, \dots, I_m \subset [n]$ be disjoint, non-empty sets such that either the union $I = I_1 \cup \cdots \cup I_m$ is not all of $[n]$ or $m \geq 3$. Then,*

$$\sum_{j=1}^m \sigma'(I_j) \leq \sigma'(I).$$

Thus if $I_1, \dots, I_m$ is a partition of $[n]$ and $m \geq 3$, then $\sigma'(I_1, \dots, I_m) \leq d$.

Proof. Any set I_j with $c_{I_j} < 1$ does not contribute to the sum on the left and so can be excluded. Doing so will decrease m, but will also ensure that I is a proper subset of $[n]$. Also, if $m = 1$, then the statement is trivial. If there exists a j such that $c_{I_j} \geq c_{[n]} - 1$, then by the disjointness, and the fact that $I \neq [n]$ or $m \geq 3$, we must have $c_{I_k} < 1$ for all $k \neq j$, and so $\sigma'(I_k) = 0$. Therefore, in this case both sides of the claimed inequality are equal to d.

Now, we are reduced to the case where $\sigma'(I_j) = \lfloor \phi(I_j, \gamma, \mathbf{c}) \rfloor + 1$ for each j. Then, we have

$$\begin{aligned} \sum_{j=1}^m \sigma'(I_j) &= \sum_{j=1}^m \left(\left\lfloor \frac{c_{I_j} - 1}{1 - \gamma} \right\rfloor + 1 \right) \leq \left\lfloor \sum_{j=1}^m \frac{c_{I_j} - 1}{1 - \gamma} \right\rfloor + m \\ &= \left\lfloor \frac{c_I - m}{1 - \gamma} \right\rfloor + m \\ &\leq \left\lfloor \frac{c_I - m}{1 - \gamma} + \frac{m - 1}{1 - \gamma} \right\rfloor + 1 = \lfloor \phi(I, \gamma, \mathbf{c}) \rfloor + 1 \end{aligned} \tag{10}$$

In addition, we can bound (10) by:

$$\begin{aligned} \left\lfloor \frac{c_I - m}{1 - \gamma} \right\rfloor + m \leq \frac{c_{[n]} - m}{1 - \gamma} + m &= \frac{c_{[n]} - m\gamma}{1 - \gamma} \\ &= \frac{d(1 - \gamma) + 1 + \gamma - m\gamma}{1 - \gamma} \\ &= d + \frac{1 - (m - 1)\gamma}{1 - \gamma} \leq d + 1 \end{aligned}$$

The first of these inequalities will be strict if $I \neq [n]$ and the second will be strict if $m \geq 3$. We conclude that $\sum_{j=1}^m \sigma'(I_j) < d + 1$, but since it is an integer it must be at most d. Therefore, $\sum_{j=1}^m \sigma'(I_j) \leq \min(\lfloor \phi(I, \gamma, \mathbf{c}) \rfloor + 1, d) \leq \sigma'(I)$ as required.

The last sentence then follows from the fact that $\sigma'([n]) = d$. □

We define $\mathcal{F}_{d,\gamma,\mathbf{c}}$ to be the set of all trees T such that each vertex v of T is either trivalent or satisfies $\sigma'(\mathrm{part}(v)) = d$, and such that for each edge e of T, $\sigma'(\mathrm{part}(e)) < d$. In other words, we define $\mathcal{F}_{d,\gamma,\mathbf{c}}$ to be the admissible collection of trees whose admissible partitions are all partitions π that either have at most 3 parts or such that $\sigma'(\pi) = d$.

Lemma 6.2. *The collection of trees $\mathcal{F}_{d,\gamma,\mathbf{c}}$ defines an admissible collection of trees.*

Proof. We show that $\mathcal{F}_{d,\gamma,\mathbf{c}}$ is an admissible collection of trees by showing that the collection of partitions is an admissible collection of partitions using Proposition 2.10. Thus, we need to check that the collection $\mathcal{P}$ of partitions π that either have at most 3 parts or have $\sigma'(\pi) = d$ satisfies the conditions of Definition 2.7. The first condition is immediate. The second condition holds because for any single index i, $\sigma'(\{i\}) = 0$ and thus $\sigma'(\{1\} \cup \{2\} \cup \cdots \cup \{n\})$ is also 0, which is less than d.

Third, suppose that $\pi \in \mathcal{P}$ and π' is a coarsening of π. If π' has 3 or fewer parts, then it is automatically in $\mathcal{P}$, so we may assume that π' has at least 4 parts, which implies that π does as well. In that case, Lemma 6.1 shows that $d \geq \sigma'(\pi') \geq \sigma'(\pi) = d$, which implies that π' must be in $\mathcal{P}$.

For the fourth condition in Definition 2.7, we suppose that $I_1, \ldots, I_l$ is a partition and that r and s are positive integers such that $r + s < l$ and both $I_1 \cup \cdots \cup I_r, I_{r+1}, \ldots, I_l$ and $I_1, \ldots, I_{l-s}, I_{l-s+1} \cup \cdots \cup I_l$ are in $\mathcal{P}$. If r or s is 1, then the condition is automatic, so we may assume that $r, s \geq 2$. Then both of the above partitions have at least 4 parts, and so Lemma 6.1 implies that

$$\begin{aligned} d &= \sigma'(I_1 \cup \cdots \cup I_r, I_{r+1}, \ldots, I_l) \\ &\leq \sigma'(I_1 \cup \cdots \cup I_r, I_{r+1}, \ldots, I_{l-s}, I_{l-s+1} \cup \cdots \cup I_l) \\ &\leq d. \end{aligned}$$

Therefore, these inequalities are equalities. In particular, by canceling, we get that $\sigma'(I_{l-s+1}, \ldots, I_l) = \sigma'(I_{l-s+1} \cup \cdots \cup I_l)$. Thus,

$$\sigma'(I_1, \ldots, I_l) = \sigma'(I_1, \ldots, I_{l-s}, I_{l-s+1} \cup \cdots \cup I_l) = d,$$

and so the partition $I_1, \ldots, I_r$ is in $\mathcal{P}$. Thus, $\mathcal{P}$ is an admissible collection of partitions, and so $\mathcal{F}_{d,\gamma,\mathbf{c}}$ is an admissible collection of trees. ☐

For a fixed set I, the set of $(\gamma, \mathbf{c})$ satisfying (9) with $\sigma'(I) < d$ is open. Thus if γ and $\mathbf{c}$ are fixed parameters and γ' and $\mathbf{c}'$ are sufficiently small perturbations, the set of admissible partitions for $(\gamma', \mathbf{c}')$ is contained in that for $(\gamma, \mathbf{c})$. Proposition 3.12 then then shows that there is a morphism $M_{\mathcal{F}_{d,\gamma',\mathbf{c}'}} \to M_{\mathcal{F}_{d,\gamma,\mathbf{c}}}$. Such morphisms are characteristic of variation of GIT [DH98, Tha96].

We say that the parameters γ and $\mathbf{c}$ are generic if $\phi(I, \gamma, \mathbf{c})$ is not an integer for any $I \subseteq [n]$.

Theorem 6.3. *Let d, γ, and $\mathbf{c}$ be parameters satisfying the conditions in (8). Let $\widetilde{M}_{\mathcal{F}_{d,\gamma,\mathbf{c}}}$ denote the normalization of the tree compactification. There is then a birational morphism $\widetilde{M}_{\mathcal{F}_{d,\gamma,\mathbf{c}}} \to U_{d,n} /\!\!/ \mathrm{SL}(d+1)$. Moreover, if γ and $\mathbf{c}$ are generic, then there is a bijective morphism $M_{\mathcal{F}_{d,\gamma,\mathbf{c}}} \to U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$.*

Proof. For convenience, we abbreviate $\mathcal{F}_{d,\gamma,\mathbf{c}}$ by $\mathcal{F}$.

First, we assume that γ and $\mathbf{c}$ are generic. In this case, the GIT quotient $U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$ is isomorphic to $M_{0,n}(Z_{\gamma,\mathbf{c}})$ by [GJM13, Thm. 5.8], where $Z_{\gamma,\mathbf{c}}$ is the extremal assignment that assigns to a tree T the set of vertices v such that $\sigma'(\mathrm{part}(v)) = d$. Here, we use the fact that for generic GIT parameters, our σ' agrees with the σ from [GJM13]. Thus, $\mathcal{F}_{Z_{\gamma,\mathbf{c}}}$ is the same admissible collection as $\mathcal{F}$, and so we have a bijective morphism $M_{\mathcal{F}} \to M_{0,n}(Z_{\gamma,\mathbf{c}}) \cong U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$ by Theorem 1.2.

Second, we suppose that γ and $\mathbf{c}$ are not generic. We still have that the varieties $M_{\mathcal{F}}$ and $U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$ are birational to each other since they are both birational to $\overline{M}_{0,n}$ by Theorem 1.1 and [GJM13, Thm. 1.1], respectively. Let Γ be the closure of the graph of this birational map in $M_{\mathcal{F}} \times U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$. We claim that the projection $\Gamma \to M_{\mathcal{F}}$ is finite, which, since this morphism is birational, will show that the normalization morphism $\widetilde{M}_{\mathcal{F}} \to M_{\mathcal{F}}$ factors as $\widetilde{M}_{\mathcal{F}} \to \Gamma \to M_{\mathcal{F}}$. Composition of the morphism $\widetilde{M}_{\mathcal{F}} \to \Gamma$ with the second projection will then give the morphism from the theorem statement.

For contradiction, we suppose that $\Gamma \to M_{\mathcal{F}}$ is not finite, which means that there exists a curve $C \in \Gamma$ that projects to a point in $M_{\mathcal{F}}$. In this case C must map to a curve in $U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$. Because we have proper birational morphisms from $\overline{M}_{0,n}$ to both $M_{\mathcal{F}}$ and $U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$, these factor through the graph to give a proper birational morphism $\psi \colon \overline{M}_{0,n} \to \Gamma$, which is thus surjective. Let C' be a curve in the preimage $\psi^{-1}(C)$ that maps surjectively onto C. Then the image $\phi_{\mathcal{F}}(C')$ is a point in $M_{\mathcal{F}}$, while the image of C' in $U_{d,n} /\!\!/_{\gamma,\mathbf{c}}$ is a curve. Let τ be the tree labelling the stratum $\overline{M}_{0,n}^{\tau}$ that meets a dense open subset of C'. As in Section 4, we use the factorization $\overline{M}_{0,n}^{\tau} \cong \prod_{v\in\tau} M_{0,\mathrm{link}(v)}$. Since $\phi_{\mathcal{F}}(C')$ is a point, the description of $\phi_{\mathcal{F}}$ restricted to $\overline{M}_{0,n}^{\tau}$ in Proposition 4.5 means that the projection of C' to a factor $M_{0,\mathrm{link}(v)}$ is a point whenever v is not an admissible vertex of τ.

Whether a curve in $\overline{M}_{0,n}$ maps to a point in the projective variety $U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$ depends only on the numerical class of the curve. Again, using the product structure $\prod_{v\in\tau} M_{0,\mathrm{link}(v)}$, the numerical class of C' can be written as a sum of classes coming from the factors. At least one of these curve classes must have positive degree in $U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$. Therefore, we can replace C' with a curve $C'' \in \overline{M}_{0,n}$, also meeting the interior of $\overline{M}_{0,n}^{\tau}$ such that the projection of C'' to $M_{0,\mathrm{link}(v)}$ is a point except for a single vertex $v \in \tau$, which must be admissible by the previous paragraph. The curve C'' thus also has $\phi_{\mathcal{F}}(C'')$ a point.

Since $M_{0,\mathrm{link}(v)}$ is not a point, v must be at least 4-valent, which, from our description of the admissible partitions in $\mathcal{F}$ means that $\sigma'(\mathrm{part}(v)) = d$. For $0 < \epsilon \ll 1$ we choose

$$c_i' = c_i + \epsilon \frac{\sigma'(I_i)}{|I_i|} \qquad\qquad \gamma' = \gamma - \epsilon \frac{d}{d-1},$$

where I_i refers to the part of $\mathrm{part}(v)$ that contains the index i. The new parameters $\mathbf{c}'$ and γ' continue to satisfy the equality in (8) because $\sigma'(\mathrm{part}(v)) = \sum_{I\in\mathrm{part}(v)} \sigma'(I) = d$.

We claim that this perturbation leaves $\sigma'(I)$ unchanged for all parts I in $\mathrm{part}(v)$. First, if $\phi(I, \gamma, \mathbf{c})$ is an integer in the range $1 \leq \phi(I, \gamma, \mathbf{c}) < d - 1$, then

$$\phi(I, \gamma', \mathbf{c}') = \frac{c_I + \epsilon\sigma'(I) - 1}{1 - \gamma + \epsilon\frac{d}{d-1}}$$

This quantity will be at least as big as $\phi(I, \gamma, \mathbf{c})$ so long as

$$\frac{\sigma'(I)}{d/(d-1)} \geq \frac{c_I - 1}{1 - \gamma} = \phi(I, \gamma, \mathbf{c}).$$

The assumption on $\phi(I, \gamma, \mathbf{c})$ implies that $c_I < c_{[n]} - 1$, so together with the integrality assumption, we have $\sigma'(I) = \phi(I, \gamma, \mathbf{c}) + 1$, and, since $\sigma'(I) \leq d$,

$$\frac{\sigma'(I)}{d/(d-1)} = \frac{\sigma'(I)d - \sigma'(I)}{d} \geq \frac{\sigma'(I)d - d}{d} = \phi(I, \gamma, \mathbf{c}),$$

which verifies the desired inequality. Thus for sufficiently small ϵ we have that $\sigma'(I)$ is unchanged. Moreover, if $\phi(I, \gamma, \mathbf{c})$ is strictly less than $d - 1$, these inequalities are strict. Second, if $\phi(I, \gamma, \mathbf{c})$ is not an integer or not in the specified range, then choosing ϵ sufficiently small will also not change $\sigma'(I)$.

Choose $\mathbf{c}''$ and γ'' to be further perturbation of $\mathbf{c}'$ and γ' so that the parameters are generic. The previous paragraph ensures that $\phi(I, \gamma', \mathbf{c}')$ is not at one of the thresholds in the definition of σ' in (9), except possibly when $\phi(I, \gamma, \mathbf{c}) = d - 1$. There is at most one such set I in $\mathrm{part}(v)$, so we can further assume that the perturbation is such that $c''_i > c_i$ for i in a set I such that $\phi(I, \gamma, \mathbf{c}) = d - 1$ if one exists. Then, $\sigma'(I)$ is again unchanged for all parts I of $\mathrm{part}(v)$, and so $\mathrm{part}(v)$ is an admissible partition of $\mathcal{F}_{d,\gamma'',\mathbf{c}''}$. Therefore, $\phi_{\mathcal{F}_{d,\gamma'',\mathbf{c}''}}(C'')$ is a point.

By the first part of the proof, and by the general theory of variation of GIT [DH98, Tha96], respectively, we have two morphisms:

$$M_{\mathcal{F}_{d,\gamma'',\mathbf{c}''}} \to U_{d,n} /\!\!/_{\gamma'',\mathbf{c}''} \mathrm{SL}(d+1) \to U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1).$$

Thus, the curve C'' maps to a point in $U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$, which contradicts our construction of C''. Therefore, we conclude that there is no positive-dimensional fiber of the projection $\Gamma \to M_{\mathcal{F}}$, so the projection must be finite, which completes the proof. $\square$

Remark 6.4. The proof of Theorem 6.3 is indirect, passing through the modular compactifications and not establishing a bijective morphism except for general GIT parameters. We expect there to be a closer relationship between the GIT quotient $U_{d,n} /\!\!/_{\gamma,\mathbf{c}} \mathrm{SL}(d+1)$ and the tree compactification $M_{\mathcal{F}_{d,\gamma,\mathbf{c}}}$.

For example, in the $d = 1$ case, the Chow variety factor of $U_{1,n}$ is trivial and $U_{1,n} \cong (\mathbb{P}^1)^n$. By the Gelfand-MacPherson correspondence [GM82], the GIT quotient of $(\mathbb{P}^1)^n$ can be recast as the GIT quotient of the Grassmannian $\mathrm{Gr}(2, n)$ by a $(n-1)$-dimensional torus. This torus acts by scaling the coordinates in the Plücker embedding in projective space, so the quotient is a subvariety of the toric variety formed by taking the quotient of projective space. Variation of GIT can be understood through this toric GIT, and this is compatible with our construction of $M_{\mathcal{F}_{1,\gamma,\mathbf{c}}}$.

7. Smooth tree compactifications

In this section we characterize smooth tree compactifications. These are special cases of the Smyth modular compactifications considered in Section 5, and have a simpler combinatorial description. The parameters in Smyth's moduli problems can allow for singularities worse than nodes and also for the marked points to coincide. However, any smooth modular compactification, and indeed, any smooth tree compactification, is isomorphic to a moduli space with only simple nodes, but possibly with coinciding points.

Thus, the smooth tree compactifications include all of Hassett's moduli spaces $M_{0,\mathbf{a}}$ of weighted stable curves of genus 0. Moreover, all smooth tree compactifications can be described by a natural generalization of Hassett's weight data $\mathbf{a} \in \mathbb{R}^n$, which is the combinatorial weight data defined below.

One consequence of passing from Hassett's weight vector to the combinatorial weight data is that the resulting moduli spaces are no longer necessarily projective. The first specific example of a non-projective Smyth modular compactification is given in [MSvAX18], which is a smooth modular compactification.

Definition 7.1. A collection S of subset of $[n]$ is called *combinatorial weight data* if S has the following properties:

(1) Any $I \in S$ has at least two elements.
(2) If $I \subset J \subset [n]$ and $I \in S$, then $J \in S$.
(3) If I is any subset of $[n]$, then either I or $[n] \setminus I$ is in S.

The combinatorial weight data S describes a moduli problem via a translation into Smyth's extremal assignment which we give below. Roughly speaking, the moduli problem described by S allows a set of marked points to coincide if and only if the set of their indices is not in S. On the other hand, a curve containing a single node is stable if also contains marked points indexed by a set in S. The standard compactification $\overline{M}_{0,n}$ corresponds to taking S to be all subsets of $[n]$ with at least 2 elements.

More generally, any weighted moduli space of genus 0 curves in the sense of [Has03] can also be described in terms of combinatorial weight data. Such a weighted moduli space is given by a real vector $\mathbf{a} \in (0,1]^n$ such that $\sum_{i=1}^n a_i > 2$. The corresponding combinatorial weight data are the sets I such that $\sum_{i\in I} a_i > 1$. However, there exist combinatorial weight data that do not come from weight vectors.

In the description of smooth Smyth compactifications in [MSvAX18], the contraction indicator is used, which is essentially the complement of the collection S. In particular, the the collection $\mathcal{C} = \{B \subset [n] : 2 \leq |B|, B \notin S\}$ is a *contraction indicator* in the sense of [MSvAX18, Definition 7.5].

Definition 7.2. An internal vertex v tree T is an *almost leaf* if it is incident to only one internal edge.

Definition 7.3. Given a combinatorial weight data S, we construct an admissible collection $\mathcal{F}_S$ consisting of all trees T such that:

(1) For each internal vertex v of T, either v is trivalent or v is an almost leaf with the set of leaves adjacent to v not in S.

(2) If e is an internal edge of T and I is the set of labels on one side of e, then either $I \in S$ or all the labels of I are attached to a single vertex.

In addition, we define an extremal assignment Z_S which assigns to a tree T, the set of vertices v of T such that there exists an edge e of T where the split of e corresponding to the subtree containing v is not in the collection S.

Proposition 7.4. *Given combinatorial weight data S, the collection $\mathcal{F}_S$ from Definition 7.3 is an admissible collection and Z_S is an extremal assignment. Moreover, the extremal assignment Z_S is the same as the extremal assignment constructed from $\mathcal{F}_S$ in the beginning of Section 5.*

Proof. We first show that Z_S is an extremal assignment. We show the second condition on extremal assignments first. Suppose that T is a tree obtained by contracting an edge e in a tree T. Let v, v' be the two vertices of e, and let $\tilde{v}$ be the vertex of T corresponding to e. We need to show that $v, v' \in Z_S(T')$ if and only if $\tilde{v} \in Z_S(T)$.

Suppose that $v, v' \in Z_S(T')$. The key point is that there must be an edge $e' \neq e$, such that one part of the split of e' contains v and v', and the set of labels of this part is not in S. If not, then the split of T' containing v and not in S must not contain v' and so must be defined by the edge e, and similarly for the split containing v' and not in S. However, then we would get that the two parts of the split at e are not in S, which contradicts the third axiom of combinatorial weight data. This edge e' then certifies that $\tilde{v}$ is in $Z_S(T)$. Conversely, if $\tilde{v} \in Z_S(T)$, then the edge e' that certifies this also gives that $v, v' \in Z_S(T')$.

To see the first condition, if T is a tree with all vertices in $Z_S(T)$, pick an internal edge e of T, and contract all other edges to form a tree T'. By the second condition on extremal assignments we have that both vertices of T' are then in $Z_S(T')$. However this means that both parts of the split at e are not in S, again contradicting the third axiom on combinatorial weight data.

Next, we show that our definitions of the admissible collection $\mathcal{F}_S$ and the extremal assignment Z_S are compatible. First suppose that T is a tree in $\mathcal{F}_{Z_S}$, and let v be a vertex in $Z_S(T)$ Since it is not adjacent to any other vertex in $Z_S(T)$, the edge whose split is not in S must be incident to v and, furthermore, must be the only non-leaf edge incident to v. In other words, v is an almost leaf, and the set of leaves incident to v must not be contained in S. By construction the vertices not in $Z_S(T)$ are trivalent, so T is in the collection $\mathcal{F}_S$ from Definition 7.3.

Conversely, for any tree T satisfying the conditions in Definition 7.3, $Z_S(T)$ consists of all the almost leaves v such that the set of labels adjacent to v is not an element of S. The vertices in $Z_S(T)$ are non-adjacent because if two almost leaves are adjacent, then there is a single internal edge, and, in that case, the third axiom of combinatorial weight data prohibits the possibility that sets of labels adjacent to both endpoints of the internal edge are not in S. Moreover, the vertices not in $Z_S(T)$ are trivalent, so T is in $\mathcal{F}_{Z_S}$. Having shown the equality of $\mathcal{F}_S$ and $\mathcal{F}_{Z_S}$, the admissibility of $\mathcal{F}_S$ follows from Proposition 5.2. □

We write M_S for the tree compactification $M_{\mathcal{F}_S}$.

Proposition 7.5. *Fix a combinatorial weight data S. The compactifications M_S and $M_{0,n}(Z_S)$ are isomorphic to each other and are smooth.*

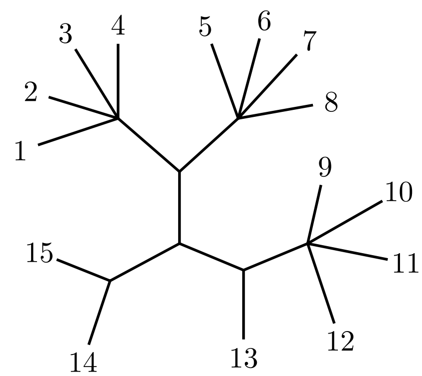

FIGURE 7. A fuzzy tree

Proof. By Theorem 1.2, M_S has a finite birational morphism to $M_{0,n}(Z_S)$ and thus, the claimed statements will follow from showing that $M_{0,n}(Z_S)$ is smooth.

We fix a $\Bbbk'$-point x in $M_{0,n}(Z)$, for an arbitrary extension $\Bbbk'$ of our field $\Bbbk$ such that x corresponds to a pointed Z-stable curve $(C, p_1, \ldots, p_n)$. By our condition on Z, the curve C is nodal and although some of the points $p_1, \ldots, p_n$ may coincide, they are all in the smooth locus of C. For each point of C with at least 1 marking, choose the minimal index, and let $A \subset [n]$ be the set of all such indices. In other words, if $i \in A$, then $p_j \neq p_i$ for $j < i$ and if $j \notin A$, then $p_j = p_i$ for some $i \in A$. In particular, C, together with the points p_i for $i \in A$, defines a stable marked curve, and thus a point in $\overline{M}_{0,A}$. Moreover, we can choose a neighborhood of x where the markings indexed by A don't coincide and continue to define a stable marked curve, therefore defining a morphism π from this neighborhood to $\overline{M}_{0,A}$.

Since the curve C and the markings p_i for $i \in A$ are determined by $\pi(x)$, each fiber of π corresponds to a choice for the points p_j for $j \notin A$. The choice of these points corresponds to a point in the $(n - |A|)$-fold product of the smooth locus of C, and therefore a neighborhood of x is isomorphic to an open subset of the $(n - |A|)$-fold fiber product of U over $\overline{M}_{0,A}$, where U is the subset of the universal family of $\overline{M}_{0,A}$ which is smooth over $\overline{M}_{0,A}$. Thus, π is a smooth morphism from a neighborhood of x to $\overline{M}_{0,A}$, which is also smooth, and so $M_{0,n}(Z)$ is also smooth. □

The converse of Proposition 7.5 is also true: the smooth tree compactifications are exactly the ones of the form M_S. We begin by giving a criterion for a tree compactification to be smooth on a standard affine open.

Definition 7.6. A tree T is a *fuzzy tree* if any non-trivalent vertices are almost leaves and are not adjacent to each other.

Proposition 7.7. *Let $\mathcal{F}$ be an admissible collection. An open affine $M_{\mathcal{F}} \cap U_T$ is smooth if and only if T is a fuzzy tree.*

Proof. We first assume that T is a fuzzy tree. We need to show that $M_{\mathcal{F}} \cap U_T$ is smooth. For each non-trivalent vertex v of T, set I_v to be the set of indices adjacent to the almost leaf v. Let S be the set of subsets of $[n]$ of size at least 2 that are not contained in any of the sets I_v. By construction, S satisfies the first two axioms of combinatorial weight data. Since the non-trivalent vertices of T are not adjacent, it is not possible for the complement of a set I_v to coincide with I_w for another vertex w. Therefore, if I is a subset of I_v, then its complement I^c is not a subset of any set I_w, and therefore I is in S, which verifies the final axiom of combinatorial weight data. Moreover, the original tree T is in the collection $\mathcal{F}_S$, so the cone C_T also occurs in the fan $\Sigma_{\mathcal{F}_S}$. These cones yield toric varieties isomorphic via an extension of the identity on the torus and thus, an isomorphism $M_{\mathcal{F}} \cap U_T \cong M_{\mathcal{F}_S} \cap U_T$. By Proposition 7.5, $M_{\mathcal{F}_S}$ is smooth and therefore $M_{\mathcal{F}} \cap U_T$ is also smooth.

We now show the converse. We show that if T is not a fuzzy tree, then there exists a birational map with target $M_{\mathcal{F}} \cap U_T$ that has an exceptional locus of codimension at least 2. If $M_{\mathcal{F}} \cap U_T$ were smooth, then van der Waerden's purity theorem [EGA4, Thm. 21.12.12] would imply that the exceptional locus is pure of codimension 1, so $M_{\mathcal{F}} \cap U_T$ must not be smooth. We first suppose that T has a non-trivalent vertex v that is not an almost leaf. We let $\mathcal{F}'$ be the collection of trees formed from $\mathcal{F}$ by replacing every tree T' that has a vertex v' with $\mathrm{part}(v') = \mathrm{part}(v)$ by the set of all maximal expansions of T' at v'. Then, $\mathcal{F}'$ is again an admissible collection of trees and there is a birational morphism $M_{\mathcal{F}'} \to M_{\mathcal{F}}$. Let τ be a tree corresponding to a stratum of $M_{\mathcal{F}'}$. Then, Proposition 4.5 implies that this stratum will be exceptional for $M_{\mathcal{F}'} \to M_{\mathcal{F}}$ if and only if τ has a vertex v' with the same partition as v. However, since the partition at v has two parts with of size at least 2, τ must have at least 2 internal edges, which means that it has codimension at least 2. Thus, $M_{\mathcal{F}} \cap U_T$ is not smooth.

Second, suppose that T has a single internal edge and both internal vertices are non-trivalent. We pick v to be one of these internal vertices and construct $\mathcal{F}'$ by replacing all trees with the same partition as at v with the expansions, in the same way as above. Again, a stratum indexed by τ is exceptional for $M_{\mathcal{F}'} \to M_{\mathcal{F}}$ if and only if a vertex of τ has the same partition as v. Suppose τ is such a tree. If τ has multiple internal edges, then the stratum has codimension at least 2. If τ has only one internal edge, then $\tau = T$ and the stratum has dimension $\deg(v) - 3$. Since the other vertex of T is adjacent to at least 3 labels, $\deg(v)$ is at most $n-2$, so this stratum also has codimension at least 2. Thus, $M_{\mathcal{F}} \cap U_T$ is again singular. □

Proposition 7.8. *If $\mathcal{F}$ is an admissible collection such that every $T \in \mathcal{F}$ is a fuzzy tree, then there exists a collection S such that $\mathcal{F} = \mathcal{F}_S$.*

Proof. We construct the set S as follows. We denote by $J_{T,v}$ the set of labels adjacent to a non-trivalent almost leaf v in a tree T. Set S to be the set of all subsets I of $[n]$ of size at least 2 such that $I \not\subseteq J_{T,v}$ for any non-trivalent almost leaf v of any $T \in \mathcal{F}$. The first two axioms for S are then satisfied by construction.

To prove that the third axiom for S is satisfied, suppose that $I \subset [n]$ has size at least 2 but is not in S. Then there is $T \in \mathcal{F}$ with $I \subseteq J_{T,v}$, so the partition with with parts $\{\{i\} : i \in I\} \cup I^c$ is admissible for $\mathcal{F}$. Sets of size at least $n-1$ are automatically in S, so $|I^c| \geq 2$. If $I^c \notin S$, then the tree T with one internal edge that has split I has the partition of both vertices admissible, so by Lemma 2.6 a contraction of T lies in $\mathcal{F}$. As the star tree is not in $\mathcal{F}$, we must have $T \in \mathcal{F}$. If $|I^c| \geq 3$ then T is not a fuzzy tree, so contradicts the assumption of the proposition. If $|I^c| = 2$ and $I^c \notin S$, then there is a tree $T' \in \mathcal{F}$ with non-trivalent vertex v with $I^c \subsetneq J_{T',v}$. Fix $i \in J_{T',v} \cap I$. Then the tree T'' with one internal edge that has split $I \setminus \{i\}$ also has both partitions admissible, so is also in $\mathcal{F}$. But then any trivalent tree with splits I and $I \setminus \{i\}$ has two contractions in $\mathcal{F}$, which contradicts $\mathcal{F}$ being an admissible collection of trees.

Finally, we show that $\mathcal{F} = \mathcal{F}_S$. The above argument shows that every partition of a tree in $\mathcal{F}_S$ is admissible for $\mathcal{F}$, so by Lemma 2.6 if $T \in \mathcal{F}_S$, then some contraction T' of T is in $\mathcal{F}$. Suppose that e is an edge of T that is contracted in T'. Since T' is a fuzzy tree, the image of e must be an almost leaf v' of T'. Then we consider the set $J_{T',v'} \notin S$, which is the split of the internal edge adjacent to v', and thus is a split of an edge e' of T. By our construction of $\mathcal{F}_S$, this means that $J_{T',v'}$ is the set of leaves adjacent to a single vertex of T, which is a contradiction. Therefore, $T' = T$. Thus $\mathcal{F}_S \subseteq \mathcal{F}$, so the second axiom of admissible collections of trees implies that $\mathcal{F}_S = \mathcal{F}$. □

Definition 7.9. Suppose that S is combinatorial data and I is a subset of $[n]$ not contained in S. We write Z_I for the closure in M_S of the stratum M_S^τ associated to the tree τ that has a single internal edge with split I.

Remark 7.10. By definition, the locus Z_I is the image of the boundary divisor in $\overline{M}_{0,n}$ corresponding to the tree τ, on which a generic point corresponds to a stable curve with two components, one marked by the elements of I and the other by the elements of I^c. Since I is not in S, such a curve is not stable according to the moduli problem associated to M_S, but we must instead contract the former component of the stable curve, meaning that points labeled by I coincide and the others are distinct. Therefore, as the closure of this stratum, Z_I parametrizes any marked stable curve where the points labeled by I coincide.

By replacing the coinciding labels with a single label, we can see that Z_I is isomorphic to $M_{0,n-|I|+1}$. In particular, it is smooth of dimension $n - |I| - 2$. This dimension can also as a consequence of the description of the dense stratum of Z_I given by Proposition 4.5.

Proposition 7.11. *If S and S' are combinatorial weight data such that $S' = S \amalg \{I\}$ for some subset $I \subset [n]$, then $M_{S'}$ is the blow-up of M_S along the smooth subvariety Z_I.*

Proof. We first prove that the toric variety $X_{\mathcal{F}_{S'}}$ containing $M_{S'}$ is an open subset of the blow-up of X_{F_S}. Indeed, to obtain $\mathcal{F}_{S'}$ from $\mathcal{F}_S$ we replace every tree $T \in \mathcal{F}_S$ that has an almost leaf v whose set of adjacent labels equals I with the trees that expand v into two extra edges, each of which ends in an almost leaf with labels I_1, I_2, where $I = I_1 \amalg I_2$. We now show that the cones $C_{T'}$ corresponding to these trees T' are cones induced by the stellar subdivision of a face of C_T.

Let J be any division of T at v. By complementing, we may assume that J is a subset of I. Then,

$$\mathbf{v}_J = \sum_{i,j \in J} \mathbf{e}_{ij} = \sum_{i,j \in J} \mathbf{v}_{\{i,j\}}.$$

Note that $\{i, j\}$ is also a valid division of T at v. Therefore, whenever J has more than two elements, it is not a ray of C_T. In other words, we can rewrite C_T as

$$C_T = \operatorname{pos}(\mathbf{v}_J : J \in \mathcal{V}) + \operatorname{pos}(\mathbf{v}_{\{i,j\}} : i, j \in I),$$

where

$$\mathcal{V} = \{\mathbf{v}_J : J \text{ is a division of } T \text{ that is not a division at the vertex } v\}.$$

Now we wish to show that the cone

$$C' = \operatorname{pos}(\mathbf{v}_{\{i,j\}} : i, j \in I).$$

is a face of C_T. The stellar subdivision of C_T using this face contains the cones $C_{T'}$ for trees $T' \in \mathcal{F}_{S'}$. Fix a division J from the collection $\mathcal{V}$. After complementing if necessary, we may assume that $I \subseteq J$. Choose $i \in I$, $j \in J \setminus I$, and $k, l \notin J$. If J is a division at an almost leaf different from v then choose j not adjacent to this vertex; this is possible because this vertex is not adjacent to v. Then by (2) we have $\mathbf{f}_{ik;jl} \cdot \mathbf{v}_J = 1$, and $\mathbf{f}_{ik;jl} \cdot \mathbf{e}_{mn} = 0$ for all $m, n \in I$. By Proposition 3.3 this functional is non-negative on C_T. Let $\mathbf{f} \in M$ be the sum of all such $\mathbf{f}_{ik;jl}$ over all choices of divisions J. Then the functional $\mathbf{f}$ is non-negative on C_T, and the face it determines is For fixed $m, n \in I$, adding the sum of all $f_{ij;kl}$ with $i, k \in I$, $j, l \notin I$ and $\{i, k\} \neq \{m, n\}$ to $\mathbf{f}$ shows that in addition $\mathbf{v}_{\{i,j\}}$ is a ray of C' and thus C_T.

Now consider a replacement tree T' corresponding to a choice $I = I_1 \amalg I_2$. Then, we have:

$$C_{T'} = \operatorname{pos}(\mathbf{v}_I : I \in \mathcal{V}) + \operatorname{pos}(\mathbf{v}_I, \mathbf{v}_{\{i,j\}} : i, j \in I_1, \text{ or } i, j \in I_2).$$

Each cone $C_{T'}$ is thus a face of the stellar subdivision of the cone C_T obtained by adding the vector $\mathbf{v}_I$. The toric variety $X_{\mathcal{F}_{S'}}$ is then an open subvariety of the blow-up of the toric variety $X_{\mathcal{F}_S}$ along the closed subvariety corresponding to the face C' of C_T.

It follows that $M_{S'}$ is the blow-up of M_S along the intersection of M_S with this torus-invariant subvariety. Set-theoretically, this intersection is equal to Z_I, and thus it just remains to show that the scheme-theoretic intersection is reduced. For this, we fix a single index $i \in I$ and consider just the rays $\mathbf{v}_{ij}$ of C' as j ranges over $I \setminus \{i\}$. Each of these rays defines a divisor in the toric variety $X_{\mathcal{F}_S}$. This divisor contains the subvariety corresponding to C', and its intersection with M_S is the divisor $Z_{\{ij\}}$. We claim that the intersection of these $Z_{\{ij\}}$ defines Z_I as a reduced scheme.

Using the modular interpretation of M_S, we can choose coordinates for an open subset of M_S meeting Z_I where the marked points i and two arbitrary points not in I are fixed at 0, 1, and ∞, respectively. In these coordinates, $Z_{\{ij\}}$ has the equation $x_j = 0$, where x_j represents the position of the marked point j. From these equations, it is immediate that the intersection of $Z_{\{ij\}}$ as j ranges over is reduced. Globally, the intersection of the the divisors $Z_{\{ij\}}$ has the expected codimension and we have shown it to be generically reduced, so it is reduced. □

Proposition 7.11 is also proved in [MSvAX18, Prop. 7.12], in the case that the ground field K has characteristic 0. One consequence of this proposition is that if S and S' are combinatorial weight data that only differ by sets of size 2, then M_S and $M_{S'}$ are isomorphic, even though they describe different moduli problems.

Example 7.12. The collection

$$S = \{I \subset [n] \mid |I| \geq 2 \text{ and } n \in I\} \cup \{[n-1]\}$$

is combinatorial weight data. Moreover, as a moduli space, M_S parametrizes collections of points $x_1, \ldots, x_n$ such that $x_i \neq x_n$ and $x_1, \ldots, x_n$ are not equal. One can check that this moduli space is isomorphic to $\mathbb{P}^{n-3}$. Any combinatorial weight data containing S can be constructed as an iterated blow-up of $\mathbb{P}^{n-3}$ and in particular, this process gives the Kapranov blow-up description of $\overline{M}_{0,n}$ [Kap93, Thm. 4.3.3].

Example 7.13. Similar to Example 7.12, we can start with the minimal combinatorial weight data

$$S' = \{I \subset [n] \mid |I \cap \{1, 2, 3\}| \geq 2\}.$$

We can think of $M_{S'}$ as the moduli space of points in $\mathbb{P}^1$ where x_1, x_2, and x_3 are fixed to be 0, 1, and ∞ respectively. Thus, $M_{S'} \cong (\mathbb{P}^1)^{n-3}$, and we get a presentation of $\overline{M}_{0,n}$ as an iterated blow-up of $(\mathbb{P}^1)^{n-3}$, which agrees with the description in [Kee92].

Remark 7.14. The connection between Kapranov's blow-up construction and variation of the moduli problem was noted by Hassett in the context of weighted moduli spaces. However, Example 7.12 is more general because there are iterated blow-ups of $\mathbb{P}^{n-3}$ which arise as tree compactifications, but not from weight data.

A nice class of examples can be found by considering the blow-ups along the toric boundary of $\mathbb{P}^{n-3}$, which are the toric varieties corresponding to graph associahedra, which themselves correspond to connected graphs on $n-2$ vertices as in [FdRJR16]. By [MSvAX18, Thm. 8.5], the toric variety of a graph associahedron is a Smyth modular compactification if and only if the complement of G is the disjoint union of complete graphs. On the other hand, [FdRJR16, Thm. 1] shows that this toric variety is a Hassett weighted moduli space if and only if the complement of G is the disjoint union of a complete graph and isolated points. Thus, if we take G to be any graph whose complement is a disjoint union of at least two non-trivial complete graphs and disjoint points, then the corresponding graph associahedron is a Smyth modular compactification and a tree compactification, but it is not a Hassett weighted moduli space. The simplest such graph is for $n = 6$ and is a cycle of with $n - 2 = 4$ vertices, whose complement is two disjoint edges, which gives an example for $n = 6$.

References

[CHMR16] Renzo Cavalieri, Simon Hampe, Hannah Markwig, and Dhruv Ranganathan, *Moduli spaces of rational weighted stable curves and tropical geometry*, Forum Math. Sigma **4** (2016), e9, 35.

[DH98] Igor V. Dolgachev and Yi Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 5–56. With an appendix by Nicolas Ressayre.

[EGA4] Alexander Grothendieck, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Quatriéme partie*, Publ. math. IHES **32** (1967).

[FdRJR16] Rodrigo Ferreira da Rosa, David Jensen, and Dhruv Ranganathan, *Toric graph associahedra and compactifications of $M_{0,n}$*, J. Algebraic Combin. **43** (2016), no. 1, 139–151.

[GJM13] Noah Giansiracusa, David Jensen, and Han-Bom Moon, *GIT compactifications of $M_{0,n}$ and flips*, Adv. Math. **248** (2013), 242–278.

[GM10] Angela Gibney and Diane Maclagan, *Equations for Chow and Hilbert quotients*, Algebra Number Theory **4** (2010), no. 7, 855–885.

[GM82] I. M. Gel′fand and R. D. MacPherson, *Geometry in Grassmannians and a generalization of the dilogarithm*, Adv. Math. **44** (1982), no. 3, 279–312.

[Has03] Brendan Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. **173** (2003), no. 2, 316–352.

[Kap93] M. M. Kapranov, *Chow quotients of Grassmannians. I*, I. M. Gel′fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 29–110.

[Kee92] Sean Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*, Trans. Amer. Math. Soc. **330** (1992), no. 2, 545–574.

[MS15] Diane Maclagan and Bernd Sturmfels, *Introduction to Tropical Geometry*, Graduate Studies in Mathematics, vol. 161, 2015.

[MSvAX18] Han-Bom Moon, Charles Summers, James von Albade, and Ranze Xie, *Birational contractions of $\overline{M}_{0,n}$ and combinatorics of extremal assignments*, J. Algebraic Combin. **47** (2018), no. 1, 51–90.

[Mum08] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin; Corrected reprint of the second (1974) edition.

[Smy13] David Ishii Smyth, *Towards a classification of modular compactifications of $\mathcal{M}_{g,n}$*, Invent. Math. **192** (2013), no. 2, 459–503.

[Tev07] Jenia Tevelev, *Compactifications of subvarieties of tori*, Amer. J. Math. **129** (2007), no. 4, 1087–1104.

[Tha96] Michael Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9** (1996), no. 3, 691–723.