# RESEARCH STATEMENT

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My research interests are in geometric analysis, and particularly in the areas of differential geometry, partial differential equations, geometric flows and geometric measure theory. I am especially interested in minimal surfaces and the mean curvature flow. In what follows, I will outline my main research and some of the results I have obtained. I will begin with a brief introduction to minimal surfaces and mean curvature flow aiming to outline the motivation for their study and their connection to other fields of mathematics.

## MINIMAL SURFACES

The theory of minimal surfaces, which are critical points for the area functional, dates back to the investigations of Euler and Lagrange, in connection with the calculus of variations, during the second half of the 18th century, and to this day remains a very active field of research. The theory became very popular during the nineteenth century with the realization of minimal surfaces as soap films by Plateau. Plateau's extensive experimentation with soap films resulted in his name being given to what is known today as *Plateau's problem*. This problem, which was first raised by Lagrange, asks for the existence of a minimal surface with a given boundary. Plateau's problem was solved (for surfaces in $\mathbb{R}^3$) by Douglas and Radó in the 1930's [36, 71]. De Giorgi, around 1960, solved the problem for general hypersurfaces [35] and, soon after, Federer and Fleming solved the problem in arbitrary dimension and codimension by the use of *integral currents* [44]. Integral currents are generalizations of oriented manifolds and provide a concept of '$k$-dimensional domains of integration in Euclidean $n$-space', which are extremely powerful tools in the calculus of variations due to their compactness properties.

The study of integral currents belongs to the field of geometric measure theory, which has been vital to the study of minimal surfaces. Apart from existence results (as in Plateau's problem) geometric measure theory has been crucial for the regularity theory of surfaces (not necessarily minimal), which is a key point in their study. One of the first and most important results in that direction is the regularity theorem of Allard [1] from the 1970's. Allard's theorem provides a way to obtain regularity of a $k$-dimensional surface in $\mathbb{R}^n$ under an assumption on the first variation of its area. This result actually holds for *weakly* defined surfaces (*varifolds*) and, moreover, extends to surfaces with boundary [2]. We note that Allard's theorem builds on a groundbreaking result of DeGiorgi [35], which covered the codimension one case (in fact, in that case, DeGiorgi provided a stronger conclusion for the singular set). These theorems introduced powerful new techniques, which have since been adapted to prove regularity theorems in various other situations, examples of which include stable minimal hypersurfaces [73], mean curvature flow [24, 92], "almost" minimal surfaces [37, 38, 65] and surfaces that minimize elliptic integrands [3, 39, 74].

Minimal surfaces have not only been studied via methods from calculus of variations but also from the point of view of differential equations. Some of the earliest work in this direction is that of Bernstein in the beginning of the 20th century, with his celebrated result asserting that the only minimal graphs over the entire plane are graphs of affine functions (meaning that the graphs are planes) [13]. Of course, any minimal surface can be written locally as the graph of a function. Due to minimality, this function must satisfy a certain quasilinear elliptic partial differential equation known as the *minimal surface equation*. Quasilinear elliptic partial differential equations have been extensively studied and their behavior is now well understood [46]. Therefore, using a local graphical representation, one can also apply the well developed tools from partial differential equations to study minimal surfaces.

The aforementioned examples hint at deep connections between the field of minimal surfaces and those of geometric measure theory and partial differential equations. These connections, in addition to the physical significance of minimal surfaces, have led to longstanding interest in their study. Indeed, further fundamental connections with minimal surfaces appear in the fields of Riemannian geometry, geometric flows and general relativity; for example, minimal surface theory played a fundamental role in the proof of the positive mass theorem [78, 79], a celebrated theorem in general relativity, and several techniques from minimal surface theory have been used in the study of mean curvature flow (see for example Huisken's monotonicity formula [56], Ecker's local monotonicity formula [40] and White's regularity theory [88, 89]).

## MEAN CURVATURE FLOW

Geometric flows have come to prominence in the last half-century due to their astonishing use in the resolution of several important outstanding problems. Mean curvature flow is the gradient flow of the area functional; it moves the surface in the direction of steepest decrease of area. It occurs in the description of the evolution of interfaces which arise in several multiphase physical models [69, 82] and has been studied in material science for almost a century to model things such as cell, grain, and bubble growth. Indeed, Mullins [69] may have been the first to write down the mean curvature flow equation. Mean curvature flow and related flows have also been used to model various other physical phenomena as well as being used in image processing.

There is an obvious connection between minimal surfaces and mean curvature flow which is the fact that the former are stationary solutions of the latter. A deeper connection appears in the study of singularities of mean curvature flow. Indeed, to understand the structure of singularities of the mean curvature flow, one performs a parabolic rescaling about a sequence of space-time points approaching the singularity. If the rescaling sequence is appropriately chosen, depending on the rate at which the curvature blows up, it will converge to a solution of mean curvature flow which moves under a one-parameter family of symmetries of the equation—either translation or dilation. Such solutions are completely determined by their shape at any fixed time, and hence singularities of the mean curvature flow are modeled by *elliptic* equations, which, moreover, closely resemble the minimal surface equation.

An important motivation for the study of mean curvature flow comes from its potential geometric applications. One reason for this is that it preserves several natural curvature inequalities, an observation that has led to its use as a means of proving deep classification theorems [26, 55, 62]. Moreover, its stationary solutions are precisely the minimal surfaces, a fact that can be used to produce the latter [26, 89]. Finally, geometric statements such as isoperimetric [80] or Alexandrov–Fenchel [8] inequalities can often be derived using geometric flows once the existence of appropriate monotone quantities is demonstrated. Indeed, Geroch's monotonicity of the Hawking mass under inverse mean curvature flow [45] was a crucial ingredient of the Huisken–Ilmanen proof of the Riemannian Penrose inequality [59] (see also [25]). The Penrose inequality remains one of the most important open problems in general relativity and the resolution of the Riemannian case was a fundamental breakthrough. The techniques of [59] will be explained in greater detail later, as they are of particular relevance to my own work: Very recently, Moore [67] and Moore and I [19] have studied variants of inverse mean curvature flow and mean curvature flow respectively in Lorentzian manifolds, with the aim to investigate marginally outer trapped surfaces (MOTS). MOTS are the Lorentzian analogue of minimal surfaces and are fundamental objects in general relativity, as their existence is related to that of black holes.

## Description of my results

*Note:* In an Appendix, I have included a brief description of the main background material needed in minimal surfaces, mean curvature flow and marginally outer trapped surfaces.

### Mean curvature flow; Ancient Solutions

An ancient solution to a geometric flow is one which is defined on a time interval of the form $I = (-\infty, T)$, where $T \leq \infty$. Ancient solutions are of interest due to their natural role in the study of high curvature regions of the flow (namely, they arise as blow-up limits near singularities) [49, 57, 58, 60, 61, 90, 91]. A special class of ancient solutions are the *translating solutions*. As the name suggests, these are solutions $\{\Sigma_t^n\}_{t\in(-\infty,\infty)}$ which evolve by translation: $\Sigma_{t+s}^n = \Sigma_t^n + se$ for some fixed vector $e \in \mathbb{R}^{n+1}$. The timeslices $\Sigma_t^n$ of a translator $\{\Sigma_t^n\}_{t\in(-\infty,\infty)}$ are all congruent and satisfy the *translator equation*, which asserts that the mean curvature vector of $\Sigma_t^n$ is equal to the projection of $e$ onto its normal bundle. Solutions of the translator equation are referred to as *translators* and we will often conflate them with the corresponding translating solution to mean curvature flow. Translating solutions arise as blow-up limits of type-II singularities, which are not well-understood [49]. Understanding ancient and translating solutions is therefore important to many applications of the flow which require a controlled continuation of the flow through singularities.

Further interest in ancient and translating solutions to geometric flows arise from their rigidity properties, which are analogous to those of complete minimal surfaces, harmonic maps and Einstein metrics; for example, when $n \geq 2$, under certain geometric conditions, the only compact, convex (or noncollapsing) ancient solutions to mean curvature flow are shrinking spheres [63, 52, 64].

In general, the question of classifying all ancient solutions is a very challenging one. There are, however, a few situations in which a satisfying answer for this question is known. For example, when the ambient space is the Euclidean plane, $\mathbb{R}^2$, the only convex, compact ancient solutions are (modulo rigid motions, time translations and paraboilic dilations) the shrinking circle and the *Angenent oval* [34], which is the family of curves $\{\Lambda_t\}_{t<0}$ defined by $\Lambda_t := \{(x, y) \in \mathbb{R}^2 : \cos x = \mathrm{e}^t \cosh y\}$. Recently, in a joint work with Langford from University of Tennessee, Knoxville and Tinaglia from King's College, London, we were able to give a new geometric proof of this result [15]. Moreover, we were able to remove the compactness hypothesis and show that in that case we get only two more solutions: the grim reaper and the stationary line.

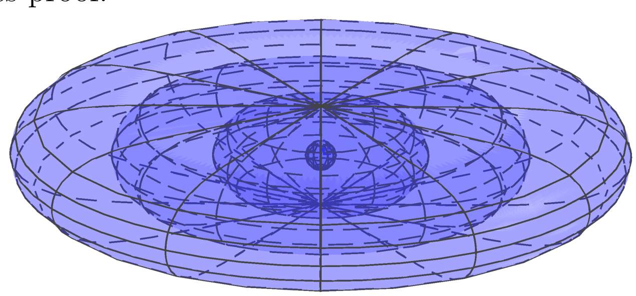
When the ambient space is a round sphere, $S^{n+1}$, the situation is particularly nice: In this case, there exists no analogue of the Angenent oval and, indeed, the only geodesically convex ancient solutions (in all dimensions) are the 'shrinking hemispheres' [27, 63].

In Euclidean ambient space of dimension $n+1 \geq 3$ the situation is a great deal more subtle as it is now known that there exist many convex ancient solutions besides the shrinking spheres. Indeed, for each $k \in \{0, \dots, n-1\}$ there exists a compact, convex ancient solution with $O(k) \times O(n+1-k)$-symmetry which contracts to a round point as $t$ goes to zero but becomes more eccentric as $t$ goes to minus infinity, resembling a *shrinking cylinder* $\mathbb{R}^k \times S^{n-k}_{\sqrt{-2(n-k)t}}$ in the 'parabolic' region and a convex, translating solution in the 'tip' region [52] (cf. [9, 91]). In particular, it sweeps out all of space.

By a theorem of Xu-Jia Wang, it is known that a compact, convex ancient solution of mean curvature flow is either entire (sweeps out all of space) or lies in a slab (the region between two parallel stationary hyperplanes) [86]. We will call entire ancient solutions with compact, convex timeslices as **ancient ovaloids** and to solutions with compact, convex timeslices which lie in slab regions as **ancient pancakes**.

Recently, Angenent, Daskalopoulos and Šešum have proved that there is only one ancient ovaloid which is uniformly two-convex and noncollapsing [10, 11]. As regard the ancient pancakes, the Angenent oval provides an example in $\mathbb{R}^2$. Existence in higher dimensions was proved by Xu-Jia Wang [86]. In a joint work with Langford and Tinaglia we have provided a different and more detailed construction of an example in $\mathbb{R}^{n+1}$ with $O(1) \times O(n)$-symmetry, including a precise description of its asymptotics [14]. In that paper, the uniqueness question was also settled: Modulo rigid motions, time translations and parabolic dilations, this solution is the only rotationally symmetric ancient pancake (note that no assumption is made concerning reflection symmetry across the $\{x_1 = 0\}$ hyperplane). Perhaps surprisingly, the precise description of the constructed solution is essential in the uniqueness proof.

Unlike the ancient ovaloids, ancient pancakes cannot possess $O(k) \times O(n+1-k)$-symmetry for any $k > 1$ (assuming the $x_1$-axis is the bounded axis) since such a solution could be enclosed by a cylinder $S^1_R \times \mathbb{R}^{n-1}$ for some sufficiently large $R$ (uniformly in time), contradicting the avoidance principle. On the other hand there should exist non-rotationally symmetric ancient pancake solutions.



The rotationally symmetric pancake shrinking to a round point.

Ancient solutions to mean curvature flow are closely related to translators, in fact the latter can be used as "building blocks" of the former. In $\mathbb{R}^3$, a complete classification of mean convex translators has only recently been obtained [4, 16, 54, 83, 86]: Spruck and Xiao showed that any mean convex translator in $\mathbb{R}^3$ is weakly convex [83]. Wang had already proved that the only convex entire example is the rotationally symmetric bowl soliton constructed by Altschuler and

Wu [4] and that any solution which is not entire necessarily lies in a slab region [86]. In a joint work with Langford and Tinaglia, we then constructed strictly convex solutions $W^n_\theta$ lying in each slab of width $\pi\sec\theta$ (and in no smaller slab) in $\mathbb{R}^{n+1}$ for any $n \geq 2$ and $\theta \in (0, \frac{\pi}{2})$ and around the same time Hoffmann et al. [54] proved that they are unique when $n = 2$ providing also different existence proofs. Since no slab of width less than $\pi$ admits a strictly convex translator (the Grim hyperplane, $\Gamma^1 \times \mathbb{R}^{n-1}$, is a barrier) and any weakly convex translator which is not strictly convex splits off a line, the only other translators, up to congruences, are the oblique products $\Gamma^2_\theta \doteqdot \sec\theta R_\theta \cdot (\Gamma^1 \times \mathbb{R})$ of Grim Reapers with lines, where $R_\theta$ is the rotation through angle $\theta$ with respect to the plane $\mathrm{span}\{e_2, e_3\}$. The solutions constructed in [16] are asymptotic to oblique Grim hyperplanes of the same width.

**Regularity estimates for almost minimal surfaces**

In the study of the geometry of surfaces in $\mathbb{R}^3$, estimates for the norm of the second fundamental form, $|A|$, are particularly powerful. In fact, when $|A|$ is bounded, the surface cannot bend too sharply and thereby estimates on the second fundamental form provide a very satisfying description of its local geometry. Moreover, such estimates allow for the use of tools from partial differential equations to study minimal surfaces, via equation (4), because of the following observation. Recall that, around any point, a surface is locally the graph of a function that satisfies (4). However, the size of the neighborhood where this graphical representation exists depends in general on the point. Estimates for the norm of the second fundamental form provide a uniform bound on the size of these neighborhoods.

When a surface is minimal $|A|^2 = -2K$, $K := k_1k_2$ being the Gaussian curvature, and such estimates are then known as curvature estimates. There are many results in the literature where curvature estimates for minimal surfaces are obtained assuming certain geometric conditions, see for instance [29, 30, 31, 53, 72, 75, 77, 84] et al.. In a joint work with Giuseppe Tinaglia of King's College London [20, 21, 22] we studied surfaces with "small" (rather than zero) mean curvature and proved various regularity results which were known to hold for minimal surfaces, such as curvature bounds and embeddedness. The "smallness" of the mean curvature is defined via a Sobolev ($W^{k,p}$) or an $L^p$ norm. Our work provides optimal conditions on the mean curvature for the results to remain true and sheds more light on the already existing estimates for minimal surfaces. The main motivation for having regularity estimates for such surfaces arises from applications in compactness arguments and perturbations of minimal surfaces.

In [20], Tinaglia and I prove estimates for the norm of the second fundamental form for surfaces with bounded *total curvature* and appropriately bounded mean curvature. In particular, we show that if the $L^2$ norm of $|A|$ is bounded, i.e. if the surface has bounded total curvature, then $|A|$ is actually pointwise bounded at interior points, provided that the mean curvature is sufficiently small in an appropriately defined norm (more specifically the scale invariant $W^{2,2}$ or $W^{1,p}$, for $p > 2$, norm). In doing this, we generalized some existing estimates for $|A|$ in the case of minimal surfaces, in particular, the renowned results of Colding and Minicozzi [30] and Choi and Schoen [29]. Furthermore, by constructing a counterexample, we show that the hypotheses on the mean curvature in our theorems are optimal.

All our results in [20] assume that the mean curvature has at least one derivative (in the *weak* sense) and, as our counterexamples show, one cannot expect curvature estimates once this assumption is removed. However, weaker conclusions can still be derived, and this is the core of our work in [22]. Inspired by the ideas of [30, 75], we prove several results on the geometry of surfaces immersed in $\mathbb{R}^3$ with small or bounded $L^2$ norm of $|A|$ and without assuming existence of any derivatives of the mean curvature. For instance, we prove that if the $L^2$ norm of $|A|$ and the $L^p$ norm of $H$, $p > 2$, are sufficiently small, then such a surface is graphical away from its boundary. We also prove that any embedded disk with bounded $L^2$ norm of $|A|$, not necessarily small, is graphical away from its boundary, provided that the $L^p$ norm of $H$ is sufficiently small, $p > 2$. These estimates, along with standard theory from partial differential equations, imply $C^{1,\alpha}$ regularity estimates.

In [21], we obtain density estimates for compact surfaces immersed in $\mathbb{R}^n$ with total boundary curvature less than $4\pi$ and with sufficiently small $L^p$ norm of the mean curvature, $p \geq 2$. In fact, we show that these estimates hold for compact branched immersions. In particular, these density estimates imply that such surfaces are embedded up to and including the boundary. Our results generalize the main results in [42] for minimal surfaces. We furthermore show how one can apply our estimates to describe the geometry and topology of such surfaces in the 2-dimensional case. More specifically, we prove that for $n = 3$ and $p > 2$ the norm of the second fundamental form $|A|$ is bounded and we use this curvature estimate to give a uniform upper bound for the genus of such surfaces. For the minimal case such results were derived in [84].

## Regularity theory for varifolds and area minimizing surfaces with boundary

### Varifolds

Regularity theory is crucial in the study of surfaces and, in particular, of minimal surfaces. A very important tool to study questions of regularity is geometric measure theory. In this field, instead of considering smooth surfaces, we consider weaker classes of objects; namely, (rectifiable) *varifolds* (which are a measure-theoretic generalization of differentiable manifolds) and *currents* (which generalize the notion of orientable manifolds). The key advantage of these objects is their simple compactness properties; for instance, one can extract a subsequential limit under the very weak assumption of uniformly bounded area. Moreover, they admit a normal almost everywhere and, consequently, the notion of the first variation of the area makes sense (as in the first equality in (3)). Finally, under certain conditions on the first variation of the area, we can define a *weak* mean curvature $H$, which is then an integrable function that satisfies the second equality in (3) (with the integration being with respect to the Radon measure associated to the varifold).

In the study of minimal surfaces (or surfaces in general) a standard technique for proving regularity results, such as curvature estimates, is that of rescaling. According to this technique, we rescale (or "zoom in") around a specific point of the surface to study the local behavior of the surface around that point. To be able then to draw a conclusion, we should be able to take a limit of these rescalings that satisfies "nice" properties. However, if we do not have good a-priori estimates, the limit might fail to exist (as a smooth manifold) and in general we will have to deal with phenomena such as a *singular set* (that is, points around which the surface is not smooth) or *multiple sheeting* (that is, two smooth pieces coming together forming what is called *multiplicity*). Studying these rescalings as varifolds allows us to attack these issues, not only because we can extract a limit under very weak conditions, but also because these general objects, by their definition, allow for both a singular set and multiplicities.

One of the most famous and remarkable regularity results, mainly due to its great generality, is Allard's regularity theorem [1]. He showed that a $k$-dimensional rectifiable varifold, under appropriate assumptions on its first variation and measure, is a $C^{1,\alpha}$ manifold, for some $\alpha \in (0,1)$. Replacing the varifold with a $k$-dimensional smooth manifold $M$ in $B_1(0)$, the unit ball in $\mathbb{R}^{n+k}$, his theorem roughly says that if the mean curvature of $M$ is in $L^p$, $p > k$, and if the area of $M$ is sufficiently close to that of a unit $k$-dimensional ball, then $M \cap B_{1/2}(0)$ is a graph of a $C^{1,\alpha}$ function with estimates, where $\alpha = 1 - k/p$. Later, Allard [2] showed that this regularity result can be extended to $k$-dimensional varifolds with a $C^{1,1}$ "boundary", i.e. to $k$-dimensional varifolds that have bounded variation away from a $(k-1)$-dimensional $C^{1,1}$ manifold $B$, which we refer to as the "boundary". Considering again the special smooth case; that is, when $M$ is a smooth $k$-dimensional submanifold of $B_1(0) \setminus B$ with $0 \in B$, Allard's boundary regularity theorem roughly says the following. If the mean curvature of $M$ is in $L^p$, $p > k$, and if the area of $M$ is sufficiently close to that of a unit $k$-dimensional half-ball, then $M \cap B_{1/2}(0)$ is a graph of a $C^{1,\alpha}$ function with estimates, where $\alpha = 1 - k/p$. These theorems are not only useful in proving regularity of varifolds, but they also provide a powerful tool for compactness theorems for smooth manifolds, as they provide the a-priori estimates needed to take a subsequential limit. Thus, it is not only the regularity that is obtained by Allard's theorems that is important, but also the fact that they provide specific $C^{1,\alpha}$ estimates (which are non-trivial even in the smooth case) that depend only on the mean curvature and the area.

Even though Allard's theorems are very general, they do require the existence of a (weak) mean curvature. Therefore, the estimates of these theorems cannot be applied to manifolds where the normal is merely Hölder continuous. Similarly, when a boundary exists, its normal needs to be differentiable almost everywhere for Allard's boundary regularity theorem [2] to apply. In my work in [17], I was able to show that Allard's boundary regularity theorem still holds in the case of $C^{1,\alpha}$ boundaries for any $\alpha \in (0,1]$. The main difficulty in considering such boundaries is that, for $\alpha < 1$, there may be no neighborhood of the boundary, $B$, on which the nearest point projection is well-defined. A key ingredient in my paper is the use of a Whitney partition of $\mathbb{R}^{n+k} \setminus B$ to define a new "distance" function that is both smooth and also "close" enough to the standard distance. It is worth noting here that in [2], even though all the estimates depend only on the $C^{1,1}$-norm of $B$, it is always assumed that $B$ is smooth, whereas in [17] this new "distance" function allows for this hypothesis to be dropped; in particular, the proof requires no higher than $C^{1,\alpha}$ regularity of the boundary.

In a more recent work [23], joint with Alexander Volkmann, we managed to also resolve the issue in the interior. In particular, we proved that Allard's regularity theorem holds for rectifiable varifolds without assuming the existence of a weak mean curvature, but instead assuming a weaker condition on the first variation of the area. This condition, which we refer to as *generalized normal of class* $C^{0,\alpha}$, is satisfied by $C^{1,\alpha}$ manifolds and is implied by the hypotheses of Allard's regularity theorem. We furthermore combined this result with the boundary regularity of [17] to include boundaries and thus provide a complete Allard-type $C^{1,\alpha}$-regularity theory for varifolds with generalized normal of class $C^{0,\alpha}$ and $C^{1,\alpha}$ boundaries.

**Area minimizing surfaces**

The Plateau problem is one of the oldest problems in minimal surface theory. It concerns the existence of an area minimizing *disk* with boundary a given curve in an ambient manifold $M$. This problem was solved in case the ambient manifold is $\mathbb{R}^3$ by Douglas [36] and Rado [71] and it was later generalized by Morrey [68] for Riemannian manifolds. In the 1980s, Meeks and Yau showed that if $M$ is a mean convex 3-manifold and $\Gamma$ is a simple closed curve in $\partial M$, then any area minimizing disk with boundary $\Gamma$ is embedded [66]. Later, White gave a generalization of this result to any genus [87]. More specifically, in the same context as in [66], he studied area minimizing surfaces of any prescribed genus (not only disks) and gave conditions for their existence and regularity. In the early 1960s, the same question was studied for absolutely area minimizing surfaces, i.e. for surfaces that minimize area among all orientable surfaces with a given boundary (without restriction on the genus). Using techniques from geometric measure theory, Federer and Fleming [44] were able to solve this problem by proving the existence of an absolutely area minimizing integral current (see also [35, 47] for the codimension 1 case; that is, the existence of a minimizing Caccioppoli set). In [76], Almgren, and Schoen and Simon showed that this current is a smooth embedded surface away from its boundary and Hardt [51] showed that it is also smooth at the boundary, provided that the prescribed boundary is smooth and lies on a convex set. Later, Hardt and Simon [50] improved this boundary regularity result by dropping the assumption that the prescribed boundary lies on a convex set.

It can be seen that there are two main versions of the Plateau problem; one of them concerns minimizing area amongst surfaces with fixed genus (area minimizing in a fixed topological class), and the other concerns minimizing area without any restriction on the genus (absolutely area minimizing case). There have also been many important results on *a-priori* bounds on the genus of an absolutely area minimizing surface bounded by a given simple closed curve [50].

Motivated by these problems, in a joint work with Baris Coskunuzer from Koç University, [18], we studied the genus of absolutely area minimizing surfaces $\Sigma$ in a compact, orientable, strictly mean convex 3-manifold $M$, with boundary, $\partial\Sigma$, a simple closed curve lying in $\partial M$. Our initial question was to examine how "large" is the class of boundary curves such that any embedded area minimizing surface that they bound has genus at least $g$, for an arbitrary given value $g$. In [18], we answered this question by showing that these curves are generic in the space of nullhomologous (i.e. that bound at least one surface) simple closed curves. This surprising theorem has a very elegant proof which is based on an operation defined on the curves in $\partial M$, which we called *horn surgery*. The horn surgery modifies the curve so that the genus of any absolutely area minimizing surface that it bounds is increased and moreover the new curve is as close as we want to the original one.

This theorem led to a series of very interesting results not only concerning area minimizing but also minimal surfaces in $M$ with boundary in $\partial M$. For instance, we show that curves that bound more than one minimal surface are generic for a strictly mean convex 3-manifold $M$. This illustrates how different the behavior of minimal and area minimizing surfaces can be: Indeed, in [33] (see also [32]), it is proven that a generic nullhomologous (or nullhomotopic, i.e. bounding a disk) simple closed curve in $\partial M$ bounds a unique absolutely area minimizing surface (or an area minimizing disk respectively) in $M$. Hence, in [18], we show that when we relax the condition of being area minimizing to just minimal, the situation is completely opposite. As previously mentioned, in [66], Meeks and Yau proved that any area minimizing disk in a mean convex 3-manifold bounded by a simple closed curve in $\partial M$ must be embedded . After establishing this result, Meeks posed the question of whether or not the same holds for stable minimal surfaces. In [48], Hall constructed an example of a simple closed curve $\Gamma$ in $S^2 = \partial B^3$, where $B^3$ is the unit 3-ball in $\mathbb{R}^3$, such that $\Gamma$ bounds a stable minimal disk $M$ in $B^3$ which is not embedded and thus answering Meeks' question in the negative. In [18], we generalize Hall's result, by showing the existence of non-embedded stable minimal surfaces (and in particular stable minimal disks) with boundary a simple closed curve in $\partial M$, for any strictly mean convex 3-manifold M.

### Null mean curvature flow and marginally outer trapped surfaces

Huisken and Ilmanen in [59] developed the theory of weak solutions for the inverse mean curvature flow of hypersurfaces in a Riemannian manifold and applied it to prove the Riemannian Penrose inequality for a connected horizon. This inequality states that the total mass of an asymptotically flat 3-manifold of nonnegative scalar curvature is bounded below in terms of the area of each smooth, compact, connected, *outermost* minimal surface in the 3-manifold. A different proof of the Riemannian Penrose inequality was given by Bray [25], which also covered the case of non-connected horizons. In 2014, Moore [67] introduced a new geometric evolution equation, the *inverse null mean curvature flow*, for hypersurfaces in asymptotically flat spacetime initial data sets, that unites the theory of marginally outer trapped surfaces (MOTS) with the study of inverse mean curvature flow in asymptotically flat Riemannian manifolds.

Recently, Moore and I [19] studied the evolution of hypersurfaces in spacetime initial data sets by their *null mean curvature*. We developed a theory of weak solutions using the level-set approach, with the aim of providing a new way of finding MOTS. The idea of using geometric evolution equations to find apparent horizons dates back to the work of Tod [85], who suggested using mean curvature flow to find MOTS in time-symmetric slices (where $K = 0$ and MOTS are minimal hypersurfaces). White [89] showed that if the initial hypersurface encloses a minimal hypersurface, the outermost such minimal hypersurface will be the stable limit of (weak) mean curvature flow. Tod, in the same paper [85], also proposed using *null mean curvature flow* in the non time-symmetric setting. Numerical results by Bernstein, Pasch and Shoemaker et al. [12, 70, 81] show convergence of the null mean curvature flow to a MOTS. Our work in [19] provides a mathematical justification of these numerical results.

Our main results in [19] can be described as follows. Let $(M^{n+1}, g, K)$ be an initial data set in a Lorentzian spacetime and consider a 2-sided closed and bounded hypersurface $\Sigma^n \subset M^{n+1}$ with globally defined outer unit normal vector field $\nu$ in $M$. Given a smooth hypersurface immersion $F_0 : \Sigma \to M$, the evolution of $\Sigma_0 := F_0(\Sigma)$ by null mean curvature is the one-parameter family of smooth immersions $F : \Sigma \times [0, T) \to M$ satisfying

$$(1) \qquad \begin{cases} \dfrac{\partial F}{\partial t}(x,t) = -(H+P)(x,t)\nu(x,t), \quad x \in \Sigma, \quad t \geq 0, \\ F(x,0) = F_0(x), \quad x \in \Sigma, \end{cases}$$

where $H := \mathrm{div}_{\Sigma_t}(\nu)$ denotes the mean curvature of $\Sigma_t := F(\Sigma, t)$ in $M$ and $P := \mathrm{tr}_{\Sigma_t} K$ is the trace of $K$ over the tangent space of $\Sigma_t$.

Analogous to the behavior of solutions to mean curvature flow, in general, singularities develop, as the null mean curvature of solutions of (1) will tend to infinity at some points. This motivates our development of a theory of weak solutions to the classical flow (1), which we implement to investigate the limit of a hypersurface moving under null mean curvature flow. To develop the weak formulation for the classical evolution (1), we use the level-set method and assume the evolving hypersurfaces are given by the level sets,

$$\Sigma_t = \partial\{x \in M \mid u(x) > t\},$$

of a scalar function $u : M \to \mathbb{R}$. Then, whenever $u$ is smooth and $\nabla u \neq 0$, the hypersurface flow equation (1) is equivalent to the following degenerate elliptic partial differential equation

$$\operatorname{div}_M\left(\frac{\nabla u}{|\nabla u|}\right) - \left(g^{ij} - \frac{\nabla^i u \nabla^j u}{|\nabla u|^2}\right) K_{ij} = \frac{-1}{|\nabla u|}.$$

We employ the method of *elliptic regularization* to solve this, and study solutions, $u_\varepsilon$, of the following strictly elliptic equation

$$\operatorname{div}\left(\frac{\nabla u_\varepsilon}{\sqrt{|\nabla u_\varepsilon|^2+1}}\right) - \left(g^{ij} - \frac{\nabla^i u_\varepsilon \nabla^j u_\varepsilon}{|u_\varepsilon|^2+1}\right) K_{ij} = -\frac{1}{\varepsilon\sqrt{|\nabla u_\varepsilon|^2+1}}. \tag{2}$$

This elliptic regularization problem sheds new light on the study of Jang's equation, as it can be interpreted as equation (6) with a gradient regularization term. Analogous to the situation for Jang's equation, the scalar term $g^{ij}K_{ij}$ obstructs the existence of a supremum estimate for a solution of (2). This is the first obstacle towards proving existence of solution $u_\varepsilon$ of (2) and, in order to overcome it, we introduce the capillarity regularization term studied by Schoen and Yau in [79]. Our main theorem in [19] roughly says that a smooth solution of (2) exists and blows up on a MOTS (for any $\varepsilon > 0$ small enough). We actually show that this blow up set is indeed the *outermost* MOTS. We additionally show that we can extract a converging subsequence of $\{\varepsilon u_\varepsilon\}$, that converges (as $\varepsilon \to 0$) locally uniformly to a limit $u$, which blows up on a *generalized* MOTS (that is, it satisfies $H + P = 0$ at points were the normal and the mean curvature are defined, which is actually something that holds almost everywhere). Then, the hypersurfaces $\Sigma_t := \{x : u(x) = t\}$ (given by the level sets of $u$) constitute the *weak* or *level set* solution of (1) and, as $t \to \infty$, they converge to a generalized MOTS . Our analysis of both the solutions $u_\varepsilon$ and the convergence $u_\varepsilon \to u$ is in many ways inspired by (and thus resembles) that of Schulze in [80], where the flow by powers of the mean curvature is studied and that of Huisken and Ilmanen [59] for the inverse mean curvature flow. For instance, we show a minimizing property for the solutions $u_\varepsilon$ that passes to the limit $u$ as a *one-sided* minimizing property. Furthermore, we show that the convergence $u_\varepsilon \to u$, is not only locally uniform, but also convergence in the sense of varifolds of the corresponding graphs. Both of these properties are inspired by similar ones in [59, 80].

Recently, in a summer research project, Ryan Unger, a senior at the University of Tennessee, Knoxville, was able to show that weak solutions $\{\Sigma_t\}_{t\geq 0}$, as defined above, actually converge to the *outermost* MOTS.

**Appendix: Background; minimal surfaces, mean curvature flow and marginally outer trapped surfaces**

Minimal surfaces

Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface with unit normal $\nu$. The differential of the normal, $A := -d\nu$, called the *second fundamental form*, gives us information on how the surface bends in space. The eigenvalues $k_1, k_2, \ldots k_n$ of $A$ are called the *principal curvatures*, $H = k_1 + k_2 + \cdots + k_n$ the *mean curvature*, and $|A| = \sqrt{k_1^2 + k_2^2 + \cdots + k_n^2}$ the *norm of the second fundamental form.* Given a vector field $X \in C_0^\infty(M)$, the space of all smooth vector fields on $M$ with compact support, one can consider the one-parameter family of variations of $M$ along $X$

$$M_{t,X} = \{x + tX | x \in M\}.$$

This gives rise to the variation of the area of $M$, which is the formula

$$\delta M(X) = \left.\frac{d\,\mathrm{Area}(M_{t,X})}{dt}\right|_{t=0} = -\int_M H\nu \cdot X. \tag{3}$$

The hypersurface $M$ is called *minimal* if it is a critical point of the area, which, by equation (3), is equivalent to the mean curvature $H$ being identically zero.

A hypersurface $M$ without boundary is called *area minimizing* if for any compact set $K \subset \mathbb{R}^{n+1}$ and any hypersurface $M'$ which agrees with $M$ outside $K$ we have $\mathrm{Area}\, M \leq \mathrm{Area}\, M'$. If $M$ does have a boundary $\partial M$, then in the aforementioned definition we additionally require that $\partial M' = \partial M$ inside $K$. A minimal surface is not in general area minimizing, as it is merely a critical point of the area functional, and to be able to deduce any minimization properties we need to consider the second variation of the area of $M$; $\delta^2 M(X) = \left.\frac{d^2\,\mathrm{Area}(M_{t,X})}{dt}\right|_{t=0}$. If a minimal surface satisfies $\delta^2 M(X) \geq 0$ for any $X \in C_0^\infty(M)$ then it is called *stable.* A stable minimal hypersurface is still not necessarily area minimizing but it does minimize area for small variations.

Minimal surfaces can be studied using tools from partial differential equations because of the following fact. When a smooth hypersurface is given by the graph of a function $u = u(x)$, with $x = (x_1, x_2, \ldots, x_n)$, which of course is true for any hypersurface locally, the mean curvature $H = H(x, u(x))$ is given by the following equation

$$\mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = H. \tag{4}$$

It is of considerable geometric interest to find solutions $u$ to (4) for a given function $H$; in this context (4) is known as the *prescribed mean curvature equation.* In the special case when $H \equiv 0$ it is known as the *minimal surface equation.* Solutions to this equation also admit a variational interpretation as minimizers of the functional

$$F(u) = \int_\Omega \sqrt{1 + |Du(x)|^2}dx + \int_\Omega \int_0^{u(x)} H(x,t)dtdx,$$

where $\Omega$ is the domain of definition of $u$, which coincides with the area functional when $H \equiv 0$. This variational interpretation provides the bridge between the study of minimal surfaces and the calculus of variations.

Mean curvature flow

Let $F_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an $n$-dimensional smooth manifold into Euclidean space. The evolution of $M_0 = F_0(M)$ by mean curvature is a smooth one-parameter family of immersions $F : M \times [0,T) \to \mathbb{R}^{n+1}$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t} F(p,t) = H(p,t)\nu(p,t) \\ F(p,0) = F_0(p), \end{cases} \tag{5}$$

where $H(p,t)$ and $\nu(p,t)$ are respectively the mean curnature and the unit normal of the hypersurface $M_t = F_t(M)$ at the point $p$, where $F_t(\cdot) = F(\cdot, t)$.

The mean curvature flow can be regarded as a *geometric heat equation*, because, in an appropriate sense, the mean curvature vector can be interpreted as the Laplacian of the immersion $F$. This resemblance to the heat equation is not merely formal; indeed, the two share many important properties. In particular, it can be shown that (5) has a unique solution for short time (at least when $M$ is compact). Furthermore, the solutions satisfy comparison principles and derivative estimates analogous to the case of parabolic partial differential equations in the Euclidean space. Finally, mean curvature flow satisfies a smoothing property: Surfaces become smoother for a short time.

However, the mean curvature flow is not really equivalent to a heat equation. For instance, in contrast to the classical heat equation, this flow is described by a nonlinear (quasilinear) system of partial differential equations and the solutions exist in general only in a finite time interval. The flow ceases to exist because of the appearance of singularities, which occur because the mean curvature blows up. Various methods have been developed to allow for the continuation of the flow beyond the *first singular time*. These are known as *weak flows*, and the most famous are the level-set flow [28, 43], the Brakke flow [24] and the flow with surgeries [62].

Marginally outer trapped surfaces (MOTS)

Let $M^{n+1}$ be a spacelike hypersurface in a Lorentzian spacetime $(L^{n+2}, h)$. By an *initial data set*, we will mean a triplet $(M, g, K)$, where $g$ is the induced metric on $M$ and $K$ its second fundamental form. Let $\vec{n}$ denote the future directed timelike unit normal vector field of $M \subset L$ and consider a 2-sided closed and bounded hypersurface $\Sigma^n \subset M^{n+1}$ with globally defined outer unit normal vector field, $\nu$, in $M$. Then, the mean curvature vector of $\Sigma$ inside the spacetime $L$, $\vec{H}_\Sigma$, is given by

$$\vec{H}_\Sigma := H\nu - P\vec{n},$$

where $H := \operatorname{div}_\Sigma \nu$ denotes the mean curvature of $\Sigma$ in $M$ and $P := \operatorname{tr}_\Sigma K$ is the trace of $K$ over the tangent space of $\Sigma$. The quantity $H + P$ corresponds to the *null expansion* or *null mean curvature* $\theta^+_\Sigma$ of $\Sigma$ with respect to its future directed outward null vector field $l^+ := \nu + \vec{n}$,

$$\theta^+_\Sigma := \langle \vec{H}_\Sigma, l^+ \rangle_h = H + P.$$

In the case $K = 0$, the *time-symmetric* case, the null mean curvature and the mean curvature coincide. The motivation for studying this quantity $\theta^+_\Sigma$ follows from the study of black holes in general relativity. Physically, the outward null mean curvature, $\theta^+_\Sigma$, measures the divergence of the outward directed light rays emanating from $\Sigma$. If $\theta^+_\Sigma$ vanishes on all of $\Sigma$, then $\Sigma$ is called a *marginally outer trapped hypersurface*, or MOTS for short. MOTS play the role of apparent horizons, or quasi-local black hole boundaries in general relativity, and are particularly useful for numerically modeling the dynamics and evolution of black holes (see for example [5, 6, 7]).

From a mathematical point of view, MOTS are the Lorentzian analogue of minimal hypersurfaces. However, since MOTS are not stationary solutions of an elliptic variational problem, the direct method of the calculus of variations is not a viable approach to the existence theory. A successful approach to proving existence of MOTS comes from studying the blow-up set of solutions of *Jang's equation*

$$\left(g^{ij} - \frac{\nabla^i w \nabla^j w}{|\nabla w|^2 + 1}\right)\left(\frac{\nabla_i \nabla_j w}{\sqrt{|\nabla w|^2 + 1}} + K_{ij}\right) = 0, \tag{6}$$

for the height function $w$ of a hypersurface. This was an essential ingredient in the Schoen–Yau proof of the positive mass theorem [79]. In their analysis, Schoen and Yau showed that the boundary of the blow-up set of Jang's equation consists of marginally trapped hypersurfaces. Building upon this work, existence of MOTS in compact data sets with two boundary components, satisfying specific geometric conditions (in particular such that the inner boundary is (outer) trapped and the outer boundary is (outer) untrapped), was pointed out by Schoen (in a talk given at the Miami Waves conference in 2004), with proofs given by Andersson and Metzger [6], and subsequently by Eichmair [41] using a different approach.

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