Matrix Algebra
Final (100pt)

"I ______________________________ will be academically honest in all of my work on this test."
PRINT NAME

X ___________________________________
SIGN NAME

No calculators are allowed. Show work for partial credit.

PART I (48pt) ALL PROBLEMS WILL BE GRADED

1. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$.
   a. **4pt** Circle the matrix product to the right that is well-defined: BC or CB.
   b. **8pt** Find $2C - A^T B$.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a matrix transformation defined by $T(\vec{x}) = \begin{bmatrix} x_1 + 2x_2 - 2x_3 \\ 2x_1 + x_3 \\ x_1 - 2x_3 \end{bmatrix}$.
   a. **6pt** Find the standard matrix to represent T.
   b. **6pt** If A is the standard matrix from part a., find $|A|$.

3. Let $rref(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
   a. **6pt** Find $rank(A)$ and $nullity(A)$.
   b. **6pt** Find a basis for $null(A)$.

4. Let $A = \begin{bmatrix} 5 & 8 \\ 1 & -2 \end{bmatrix}$ be in $\mathbb{R}^{2x2}$. And, note the eigenvectors of A are $\begin{bmatrix} 8 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
   a. **12pt** If $P = \begin{bmatrix} 8 & -1 \\ 1 & 1 \end{bmatrix}$, find $P^{-1}AP$.

PART II (40pt) CIRCLE THE FOUR (4) PROLEMS YOU WISH TO BE GRADED

5. **8pt** Let A be in $\mathbb{R}^{4x4}$ and E_i be an elementary matrix for each i. Consider $E_1 \cdots E_{10} A$. Note that multiplication by E_3 has the effect of $4r_2 \mapsto r_2$, multiplication by either E_1 or E_7 has the effect of a row operation of the form $r_i + cr_j \mapsto r_i$, and multiplication by all other E_i has the effect of a row swap. If $|A| = 3$, find $|E_1 \cdots E_{10} A|$.

6. Let V be the space of symmetric matrices in $\mathbb{R}^{4x4}$.
   a. **4pt** Find the dimension of V.
   b. **6pt** If $\vec{v}_1, \ldots, \vec{v}_{12}$ are in V, could these vectors be linearly independent? Explain.

7. **10pt** Let A be in $\mathbb{R}^{3x3}$. A has an eigenvector of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ with an associated eigenvalue of 5, A has an eigenvector of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with an associated eigenvalue of 3, and A has an eigenvector of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with an associated eigenvalue of -7. Find A.

8. **10pt** Consider $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Do these four vectors span $\mathbb{R}^3$? Show/explain why or why not.

9. Let A be an invertible matrix in $\mathbb{R}^{nxn}$, and let $\vec{c}_1, \ldots, \vec{c}_n$ be the columns of A.
   a. **6pt** A cannot have an eigenvalue of zero. Explain.
   b. **4pt** If $A\vec{x} = \vec{b}$ and $B = \{\vec{c}_1, \ldots, \vec{c}_n\}$, then $\vec{x} = [\vec{b}]_B$. Explain.

10. Let $|a| \le 1$ and $A = \begin{bmatrix} a & -\sqrt{1-a^2} \\ \sqrt{1-a^2} & a \end{bmatrix}$.
   a. **6pt** Verify the columns of A form an orthonormal basis of $\mathbb{R}^2$.
   b. **4pt** If $\vec{x}$ is in $\mathbb{R}^2$ and $\|\vec{x}\| = 3$, then $\|A\vec{x}\| = 3$. Explain.

PART III (12pt) CIRCLE THE TWO (2) PROBLEMS YOU WISH TO BE GRADED

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{0}$	$\overline{1}$

$\times$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{1}$

11. Let $F = \{\overline{0}, \overline{1}, \overline{2}\}$ have the following addition and multiplication tables.
    a. **2pt** There are 81 matrices in F^{2x2}. Explain.
    b. **4pt** Show that there are 48 invertible matrices in F^{2x2}. (Hint: lin ind of cols?)

12. **6pt** Find an orthonormal basis of $\mathbb{R}^3$ containing $\begin{bmatrix} 6/7 \\ 3/7 \\ 2/7 \end{bmatrix}$ and $\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$.

13. **6pt** Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be a solution to $x^2 + y^2 = r^2$, where r is some real number. Also, let A be the matrix from problem #10 (of PART II). $A\begin{bmatrix} x \\ y \end{bmatrix}$ is also a solution to $x^2 + y^2 = r^2$. Show/explain why.

14. Let A be in $\mathbb{R}^{nxn}$, and let C_{ij} denote the cofactor associated with the ij entry.
    a. **3pt** The expression $a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn}$ equals zero when $i \neq k$. Explain. (Hint: duplicate row?)
    b. **3pt** Show/explain why $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & |A| \end{bmatrix}$.

15. Let A and B be in $\mathbb{R}^{nxn}$.
    a. **3pt** If AB equals the zero matrix, then every column vector of B is orthogonal to every row vector of A. Explain.
    b. **3pt** If A is noninvertible, then a nonzero matrix B in $\mathbb{R}^{nxn}$ such that AB is the zero matrix necessarily exists. Explain. (Hint: fund thm?)