CURVATURE FLOW OF NETWORKS
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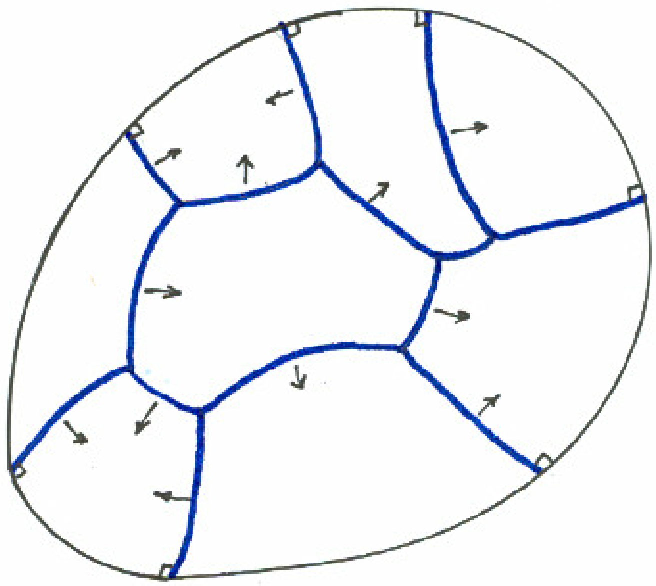


$D \subset \mathbb{R}^2$ smooth, convex

edges move with normal velocity= curvature

orthogonal intersection at ∂D

Plateau angles ($120°$) at triple junctions

TWO BASIC FACTS

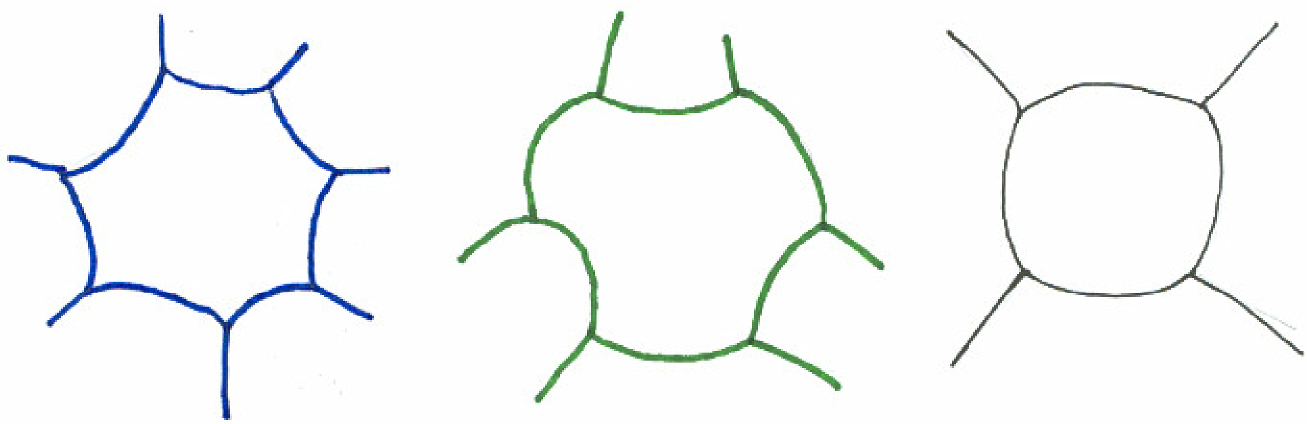
1. The flow decreases total edge length (L):

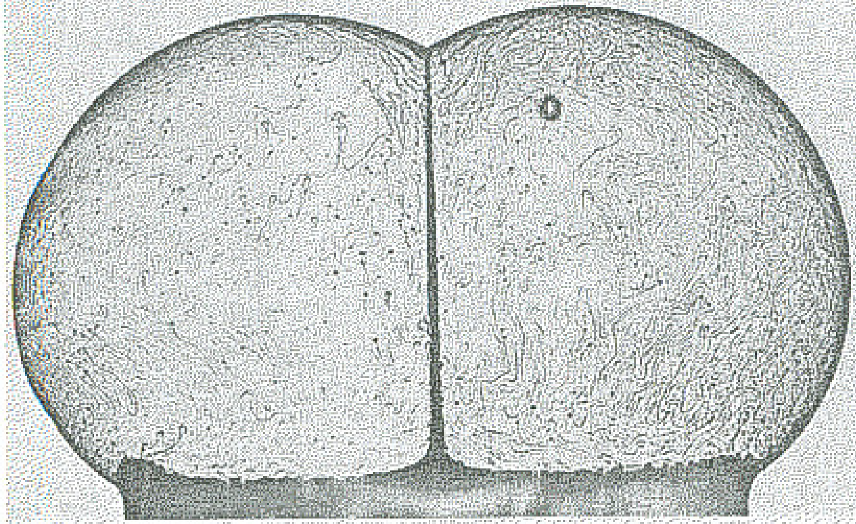
$$\gamma_t = kN + vT$$
$$\frac{dL}{dt} = -\sum_e \int_e k^2 ds$$

2. Gauss-Bonnet and the area enclosed by a cell (A):

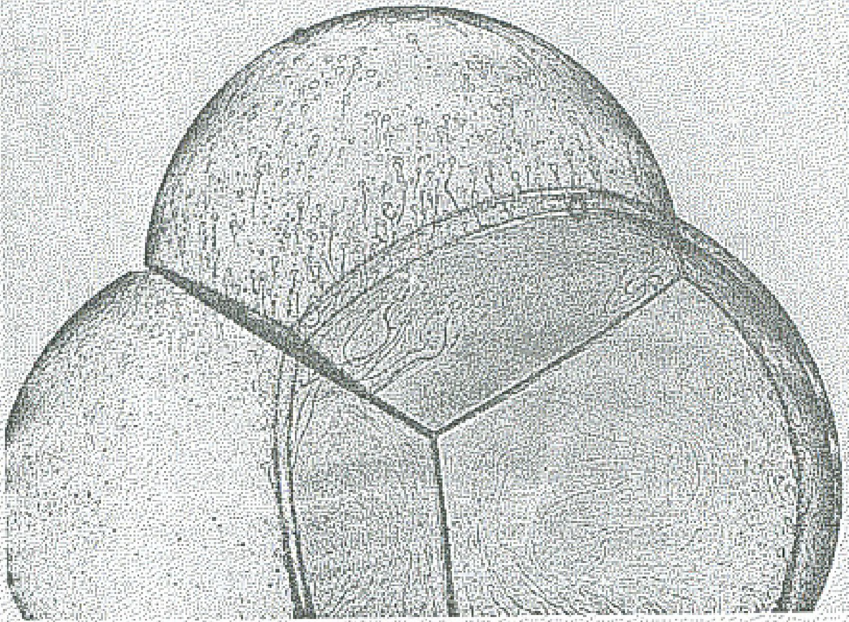
$$\frac{dA}{dt} = -\int_{\partial C} k\ ds = (n-6)\frac{\pi}{3},$$

assuming Plateau angles. Thus the area of a cell *expands* if $n > 6$ ($k < 0$ if constant sign), is *constant* if $n = 6$ ($k = 0$)and *contracts* if $n < 6$ ($k > 0$).

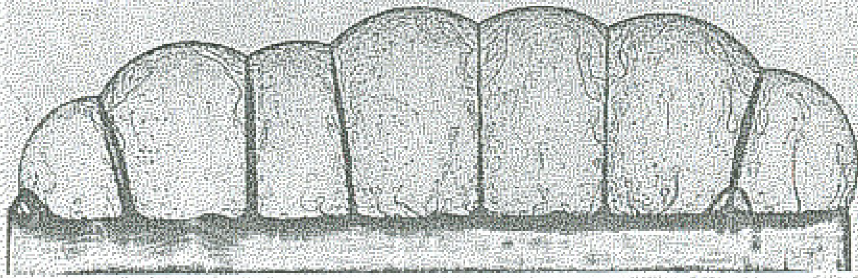




Two bubbles on a plate.



Four bubbles.



A "worm" of seven bubbles.

ALLEN-CAHN REGULARIZATION

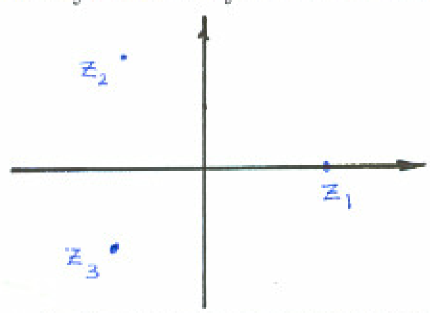
(L.Bronsard, F.Reitich, *ARMA* 1983)

$$u : D \times [0, T) \to \mathbb{R}^2$$

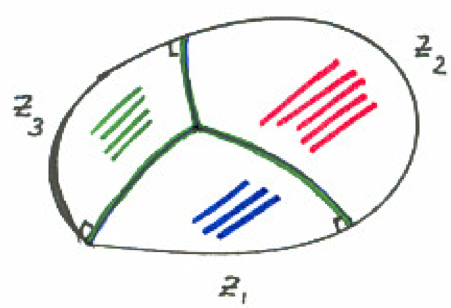
$$u_t = \epsilon^2 \Delta u - \text{ grad } W(u) \quad 0 < \epsilon << 1.$$

$$\frac{\partial u}{\partial n}|_{\partial D} = 0, \quad u|_{t=0} = u_0.$$

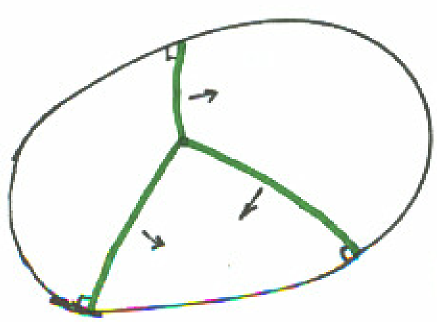
$W : \mathbb{R}^2 \to \mathbb{R}$ with 3 symmetrically located minima; e.g. $W(z) = |z^3 - 1|^2$.



original time scale(fast)$\to$ phase separation and Plateau angles



slow time scale ($\tau = \epsilon^2 t$)$\to$ interfaces move by curvature (surface tension $\to$ perimeter minimization).

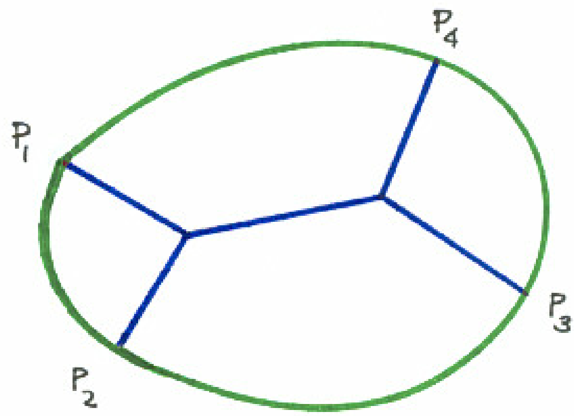


LINEARIZED STABILITY OF EQUILIBRIA

R.Ikota, E.Yanagida- *Calc. Variations 2005*- for networks without cells (trees)

Simplifying assumptions:
m boundary points $P_1, \ldots, P_m$.
∂D convex ($k(P_i) > 0$).

Definition: $I = L - \sum_{i=1}^{m} \frac{1}{k(P_i)}$



Theorem (Ikota-Yanagida): (i) The linearized unstable dimension is given by:

$$N_u = m \text{ if } I > 0; \quad N_u = m - 1 \text{ if } I \leq 0;$$

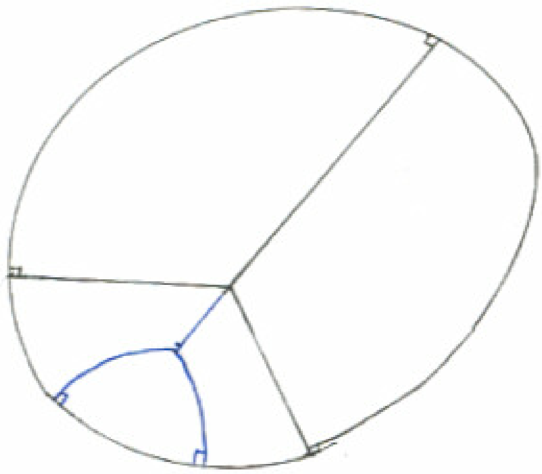
(in particular, convex domains do not support stable networks.)

(ii) (Degeneracy) A zero eigenvalue occurs if and only if $I = 0$

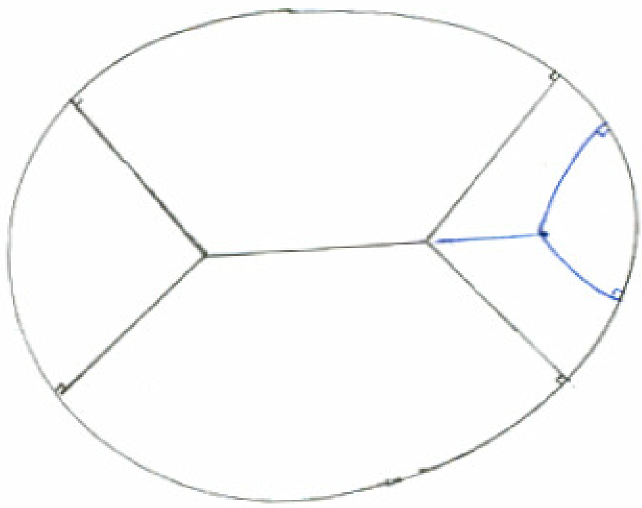
Remark: The total length L of an equilibrium Plateau network in a convex domain depends *only* on the position of the boundary points.

STATIC CONFIGURATIONS IN CONVEX REGIONS

(STABILITY)



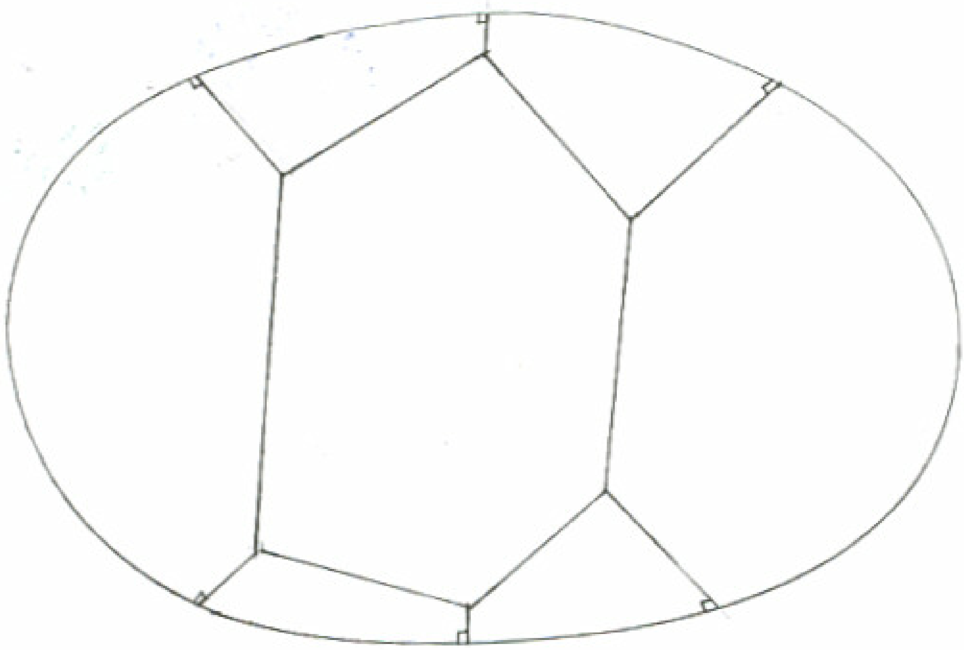
(2 unstable modes)



(3 unstable modes).

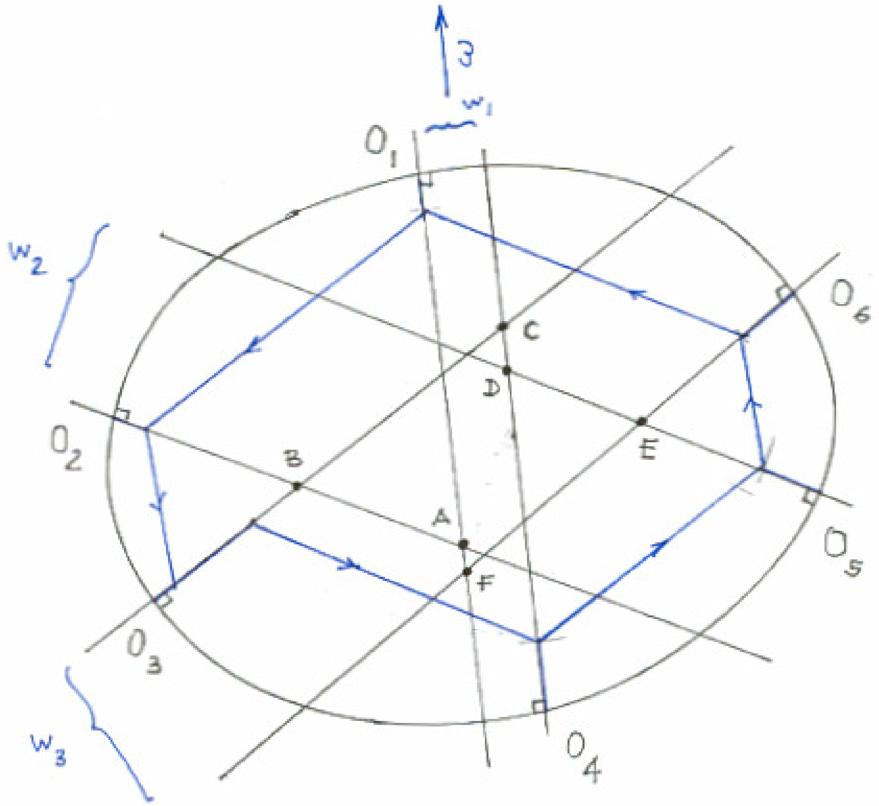
existence ?

lengths ?



(centrally symmetric)

EXISTENCE



"holonomy problem"

1. For centrally symmetric domains, there is no 'holonomy'

2. For general (smooth, convex) domains, there is always at least one direction w/ zero holonomy

(the holonomy depends only on the boundary points)

3. For generic domains, only a finite number of boundary configurations yield zero holonomy (rigidity).

Rk. the total length depends only on the boundary points.

CURVATURE MOTION OF TRIPLE JUNCTIONS

$D \subset \mathbb{R}^2$ with smooth, strictly convex boundary

$$\gamma^1, \gamma^2, \gamma^3 : I \times [0,T) \to D$$

(i) Curvature flow:

$$\langle \gamma^i_t, \quad N^i \rangle = k^i$$
$$\gamma_t = kN + vT$$

(ii) Plateau conditions:

$$\gamma^i \text{ meet at } J(t), \text{ with } 120^\circ \text{ angles}$$

(iii)Boundary conditions:

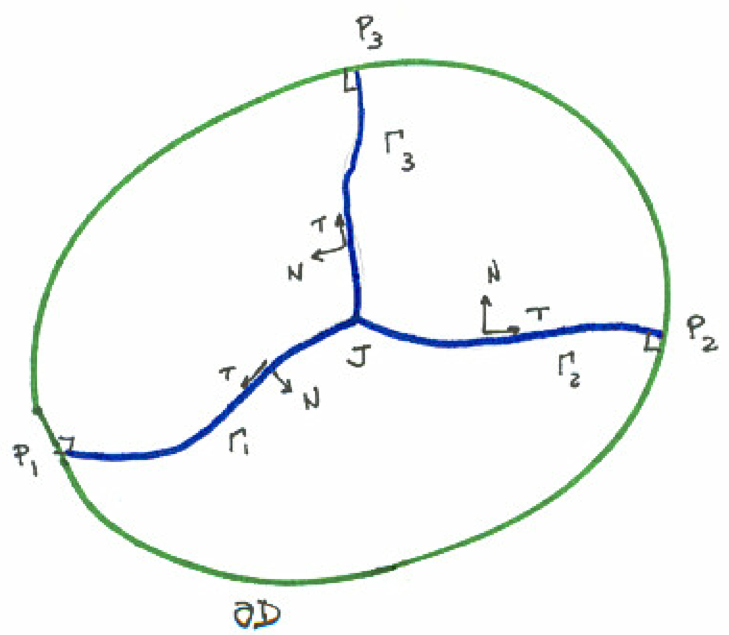
$$P, Q, R \text{ fixed (Dirichlet)}$$

OR

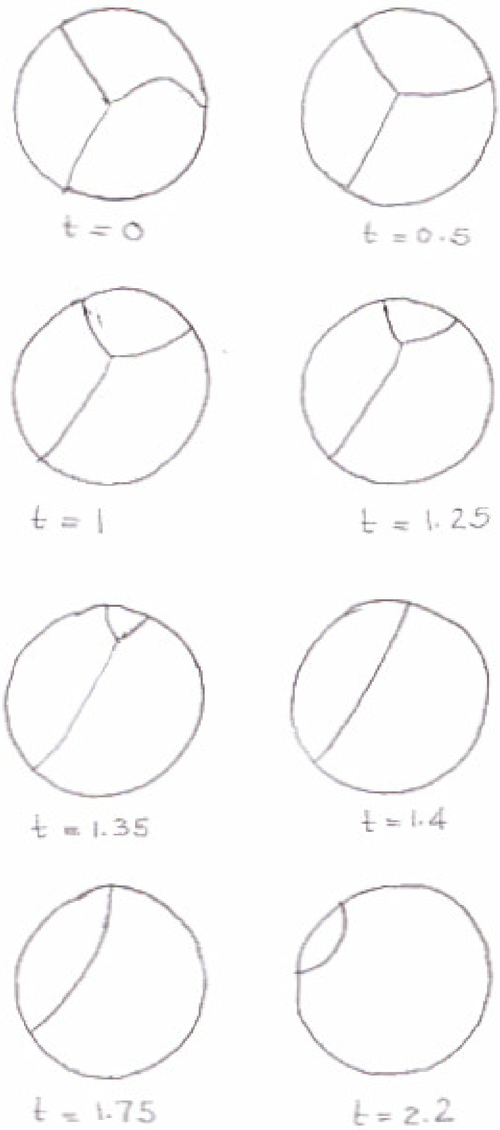
$$\gamma^i \text{ intersects } \partial D \text{ orthogonally (Neumann)}$$

(iv) Initial condition:

$$\gamma^i|_{t=0} = \gamma^i_0$$



BRONSARD / WETTON



PARAMETRIZED FLOW

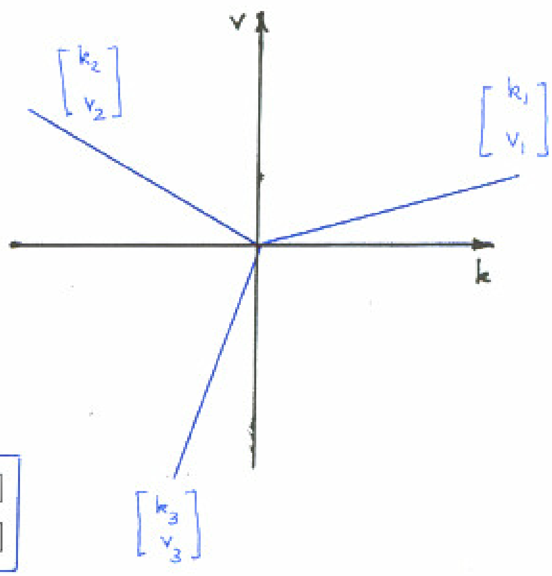
$$\gamma^i : [0,1] \times [0,T] \to D \quad i = 1,2,3.$$

$$\boxed{\gamma^i_t = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2}}$$

$$\begin{cases} \gamma^1(0,t) = \gamma^2(0,t) & = \gamma^3(0,t) \\ \langle T^i, T^j \rangle|_{(0,t)} & = -\frac{1}{2} \end{cases}$$

$$\begin{cases} b(\gamma^i(1,t)) & = 0 \qquad \partial D = \{b = 0\} \\ \langle \nu(\gamma^i), N^i \rangle_{(1,t)} & = 0 \qquad \nu = \text{ inner unit normal to } \partial D \end{cases}$$

$$\gamma^i(x,0) = \gamma^i_0(x).$$



Compatibility conditions

$$\gamma_t = kN + vT$$

At the junction:

$$\boxed{\begin{aligned} [k_2, v_2] &= R_{2\pi/3}[k_1, v_1] \\ [k_3, v_3] &= R_{2\pi/3}[k_2, v_2] \end{aligned}}$$

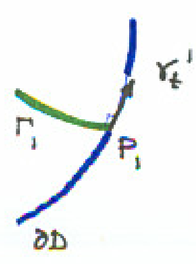
In particular, $\sum_i k_i = 0$ and $\sum_i v_i = 0$.

$$k_{1s} + k_1 v_1 = k_{2s} + k_2 v_2 = k_{3s} + k_3 v_3 = \Lambda$$

At the boundary:

$$v_i|_{x=1} = 0$$

$$k_{is} = k_i \kappa(P_i) \qquad \kappa : \text{ boundary curvature}$$



LOCAL EXISTENCE RESULTS

[Bronsard-Reitich, Neumann BC]: Compatibility conditions hold at $t = 0 \Rightarrow$ local well-posedness in $C^{2+\alpha}$.

[Mantegazza-Novaga-Tortorelli (*Annali SNS Pisa 2005*), Dirichlet BC]:

(i) Unique evolution up to reparametrization;

(ii) The solution exists as long as $\int_{\mathbb{T}} k^2 ds$ is bounded above and the lengths L^i are bounded below;

(iii) The solution remains embedded; the edges do not touch the boundary before the junction;

(iv) Isoperimetric inequality- the quantity:

$$E(t) = \sup_{p,q \in \mathbb{T}} \frac{|p-q|^2}{A_{p,q}}$$

is monotone increasing (if at least one minimizing pair does not coincide with the junction or include a boundary point.)

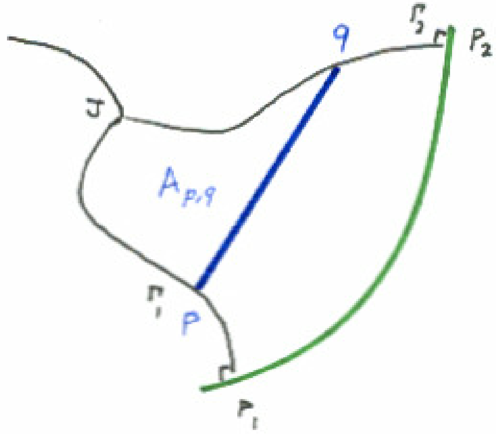
(v) Higher regularity (assuming higher order compatibility conditions at $t = 0$.)

(vi) No 'Type I' curvature blowup in finite time:

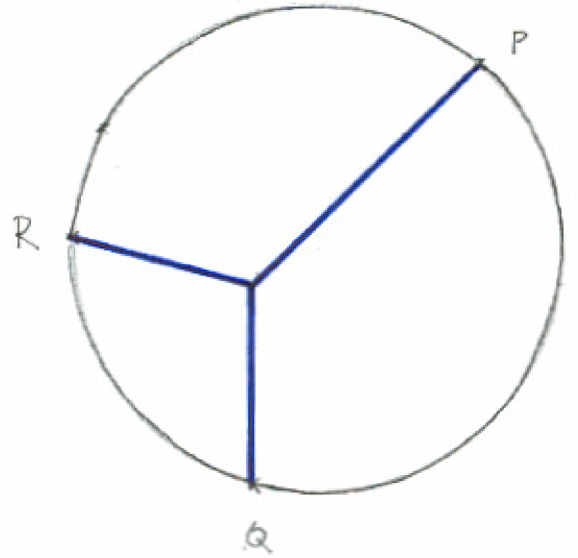
$$\text{Type I: } \lim_{t \to T_c} \sup_{\gamma^i} |k| = \infty \text{ for some i and } k \leq \frac{C}{\sqrt{T_c - t}}$$

(Not able to rule out 'Type II' curvature blowup.)

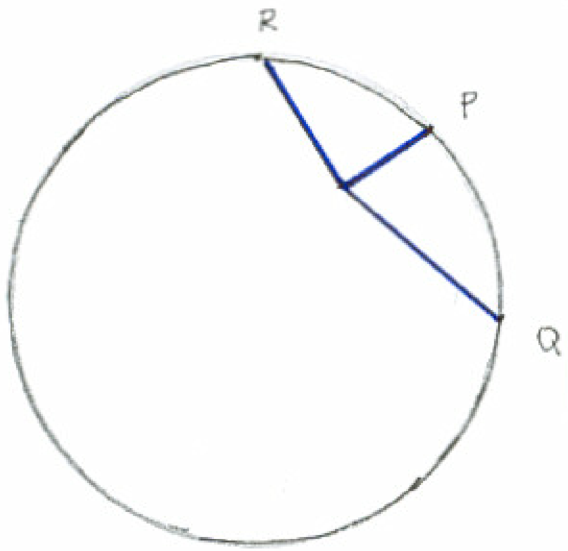
Conjecture [M-N-T]: the Dirichlet solution converges in finite time to the unique minimal length connection with the same endpoints, when one exists (otherwise the junction moves to the boundary).



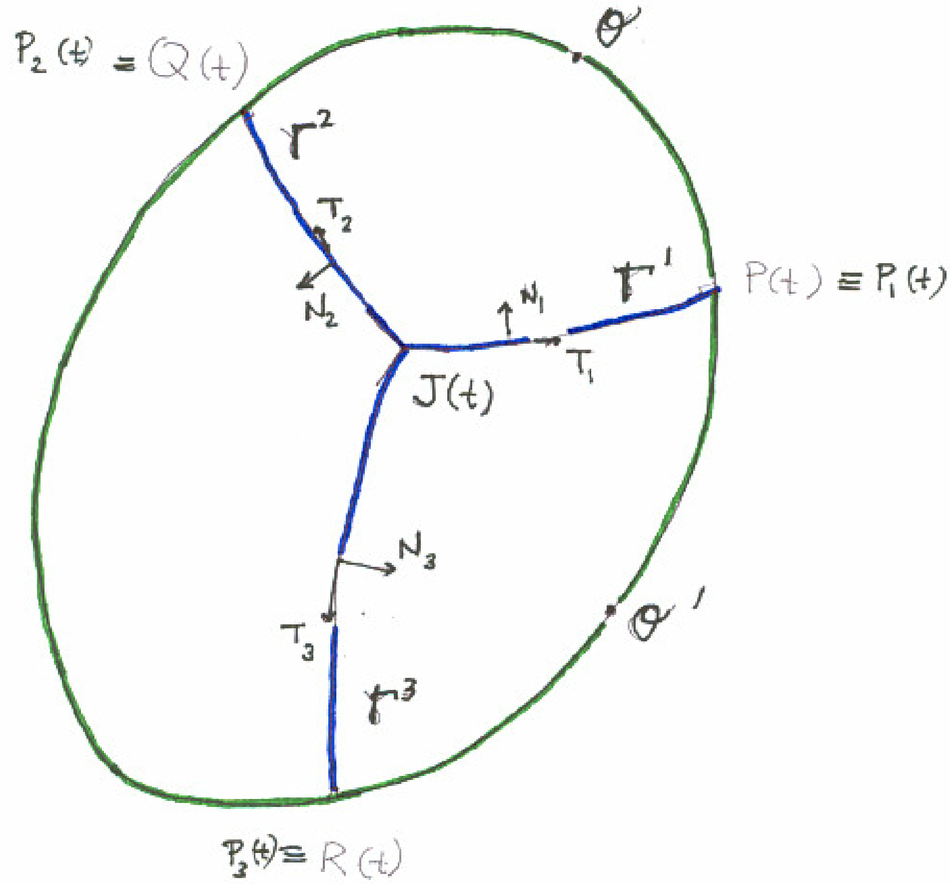
DIRICHLET PROBLEM



minimizer



no minimizer



Triple junctions with convex arcs

$D \subset \mathbb{R}^2$ smooth convex

$k^1 > 0$, $k^2 < 0$, $k^3 > 0$

Γ^i intersect ∂D orthogonally (Neumann)

Γ^i meet at $J(t)$ w/ $120°$ angles

TRIPLE JUNCTIONS IN THE CONVEX SETTING

$D \subset \mathbb{R}^2$ smooth, convex.

$k_1 > 0, \quad k_2 < 0, \quad k_3 > 0$

Neumann BC at ∂D, Plateau conditions at $J(t)$.

GOAL: A priori estimates of k^i in terms of 'first-order quantities' (lengths L_i, enclosed area, etc.) We'd like to show that, for a 'convex' triple junction, the maximum curvature $\sup_{\mathbb{T}}$ cannot 'blow up' before the junction hits the boundary (a form of *global existence*)

Equivalent reaction-diffusion systems

$$\mu(x,t) := \frac{1}{|\gamma_x|}, \quad (v = -\mu_x).$$

$$\left\{ \begin{array}{rcl} \mu_t &=& \mu^2 \mu_{xx} + k^2 \mu \\ k_t &=& \mu^2 k_{xx} + k^3 \end{array} \right. \tag{1}$$

Remark: a proof of the local existence/continuation theorem could be based on this system.

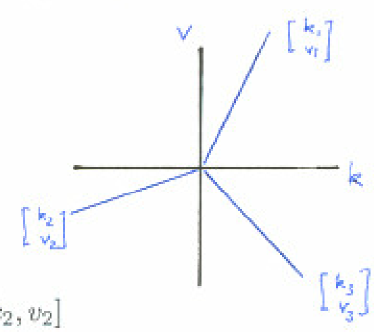
$$\left\{ \begin{array}{rcl} \mu_t &=& \mu_{ss} + v\mu_s + k^2 \mu \\ k_t &=& k_{ss} + v k_s + k^2 \end{array} \right. \tag{2}$$

$f_s := \mu(x,t) f_x$ (time-dependent operator: $f_{st} \neq f_{ts}$)

$$\left\{ \begin{array}{rcl} m_t &=& m_{ss} + d m_s + c^2 m \\ c_t &=& c_{ss} + d c_s + c^3 \end{array} \right. \tag{3}$$

arc-length coordinates: $s = \int_0^x \mu^{-1}(\xi,t)d\xi; \quad c(s,t) = k(x(s,t),t); \quad c_{st} = c_{ts}$.

$$d(s,\cdot) = \int_0^s c^2 d\sigma + v|_J$$



BC at J (for system (2)):

$$[k_2, v_2] = R_{2\pi/3}[k_1, v_1], \qquad [k_3, = v_3] = R_{2\pi/3}[k_2, v_2]$$

BC at ∂D:

$$v = 0, \qquad k_s = k\kappa$$

PRESERVATION OF CONVEXITY

Proposition. The conditions $k_1 > 0$, $k_2 < 0$, $k_3 > 0$ are preserved(if they hold at $t = 0$.)

This implies: $v_1 > 0$, $v_3 < 0$ and $|v_2/k_2| < 1/\sqrt{3}$ at the junction, for all t.

Idea of proof for k_1: 'minimum' principle. k_1 solves:

$$k_t = k_{ss} + vk_s + k^3.$$

(i) k_1 does not vanish for the first time in the interior of Γ_1, or at the boundary of the domain (since $k_{1s} > 0$ there. If it vanishes at all, this must happen first at the junction J (and then $k_{1s} > 0$ there).

(ii) At J, $k_1 = -(k_2 + k_3)$. The sum of the curvatures solves:

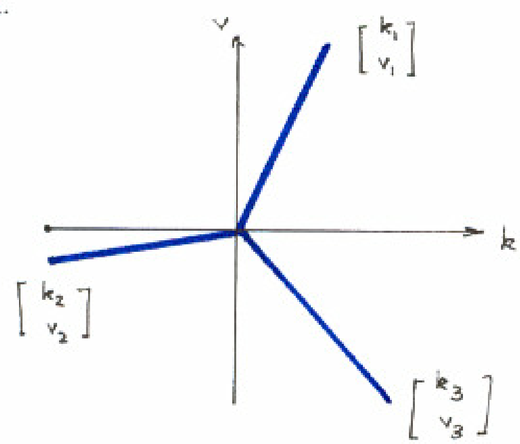
$$(c_2 + c_3)_t = (c_2 + c_3)_{ss} + d_2(c_3 + c_2)_s + (d_3 - d_2)c_{3s} + (c_2 + c_3)h,$$

where $h \geq 0$. One can show that $c_2 + c_3$ satisfies a maximum principle (close to J).

(iii) $c_2 + c_3 < 0$ initially, and if it vanishes it must first do so at J, with $(c_2 + c_3)_s < 0$ there. But the boundary conditions at J imply that, at that moment:

$$c_{1s} = (1/2)(c_2 + c_3)_s,$$

a contradiction.



SUPPORT FUNCTIONS

For well-chosen 'origins' O and O' (points on ∂D), we define:

$$p_1 = \langle \gamma^1_O, -N^1 \rangle; \quad p_2 = \langle \gamma^2_O, N^2 \rangle; \quad p_3 = \langle \gamma^3_{O'}, -N^3 \rangle > 0,$$

all positive functions along each of the arcs Γ_i. Choose O and O' so that (with i=1,2 for O, i=3 for O'):

(S1) Γ_i, ∂D and the line segment from O to J bound a convex subset of D;

(S2) $\langle \gamma^2_O, T_2 \rangle|_J < 0$ (*or* $\langle \gamma^1_O, T_1 \rangle|_J < 0$) and $\langle \gamma^3_{O'}, T_3 \rangle|_J < 0$.

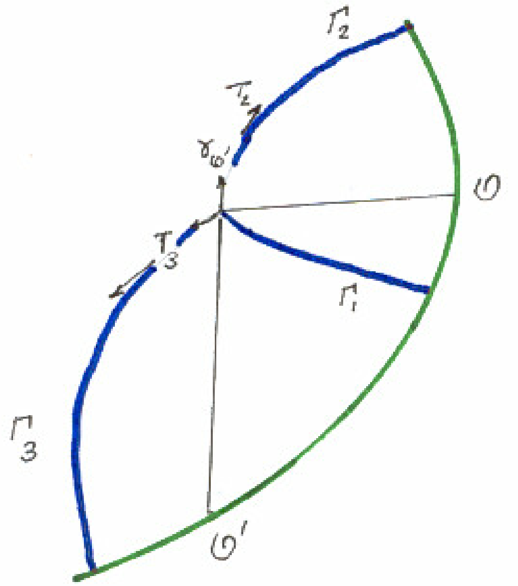
In arc length coordinates, k and p evolve according to:

$$\begin{aligned} k_t &= k_{ss} + k_s \lambda + k^3 \\ p_t &= p_{ss} + p_s \lambda + k^2 p - 2k \end{aligned}$$

It is not hard to see that p is bounded below on Γ_i; in fact: $\min_{\Gamma_i} p = dist(O, \Gamma_i)$.

Assume $p \geq 2b > 0$ on Γ_i and let $f = \frac{k}{p-b}$. The 'ratio' f satisfies the differential inequality (cp. A.Stahl 1996):

$$f_t - f_{ss} \leq (-b^2 f + 2) f^2 - \frac{2p_s}{p-b} f_s.$$



CURVATURE ESTIMATES

(D = unit disk)

Main Lemma   Unless $f(\cdot,t)$ attains its maximum only at the junction, we have: (on Γ_1 and Γ_2)

$$\sup_{\Gamma} |k| \le C \max \left\{ \frac{1}{d^2(\mathcal{O},\Gamma)}, \frac{1}{d(\mathcal{O},\Gamma)} \left(1-\cos\frac{L}{2\pi}\right) \right\} := \mathcal{M}_{\mathcal{O}}$$

(In particular, $|k|$ is bounded on Γ as long as $d(\mathcal{O},\Gamma)$ is bounded below.

Technical lemma   at the junction: $\left|\frac{v_2}{k_2}\right| \le \frac{1}{\sqrt{3}}, \left|\frac{v_1}{k_1}\right| \le \frac{1}{\sqrt{2}} \quad \forall t.$

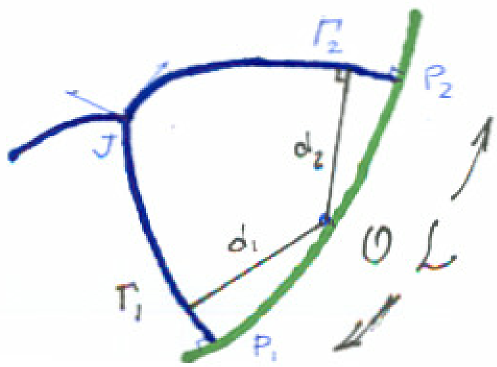
Cases

(i) $|f_2|$ max away from $J \Rightarrow \sup_{\Gamma_2} |k_2| \le \mathcal{M}_{\mathcal{O}} \Rightarrow k_1|_J \le \mathcal{M}_{\mathcal{O}} \Rightarrow \sup_{\Gamma_1} |k_1| \le \mathcal{M}_{\mathcal{O}}$

(ii) $|f_1|$ max away from $J \Rightarrow \sup_{\Gamma_i} |k| \le \mathcal{M}_{\mathcal{O}} \quad i = 1,2$

(iii) $|f_1|$ and $|f_2|$ max away from J : not possible (using (S2) & tech. lemma)

CONCLUSION : lower bound on $\boxed{\mathcal{L}(t)} \Rightarrow$ upper bound on $\boxed{\sup_{\Gamma_1 \cup \Gamma_2} |k| \ (t)}$



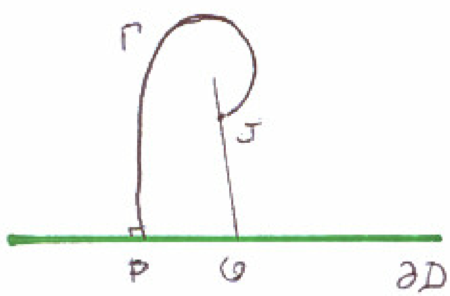
What about $\boxed{\sup_{\Gamma_3} k_3}$?

(a) k_3 attains its max at $J \Longrightarrow \sup_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} |k| \leq c\, \mathcal{M}_{\mathcal{O}}(t)$

(b) $\exists\, \mathcal{O}'$ satisfying (S1) $\Longrightarrow \sup_{\Gamma_3} k_3 \leq \mathcal{M}_{\mathcal{O}'}(t)$.

otherwise, we cannot control $\sup_{\Gamma_3} k_3$.

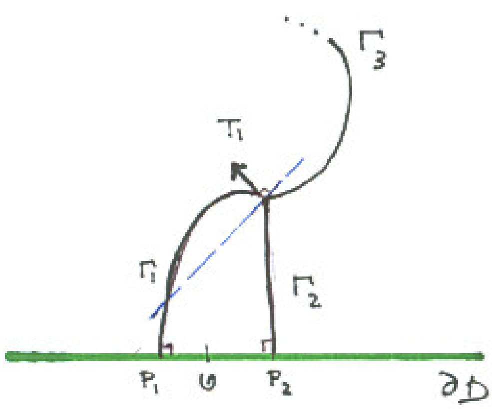
PROBLEM $\mathcal{O}$ satisfying (S1), (S2) doesn't always exist!



(S1) fails for any $\mathcal{O} \in \partial D$

condition : $k_\Gamma := \int_\Gamma k \, ds \leq \pi$

$\Longrightarrow \exists\, \mathcal{O} \in \partial D$ satisfying (S1)

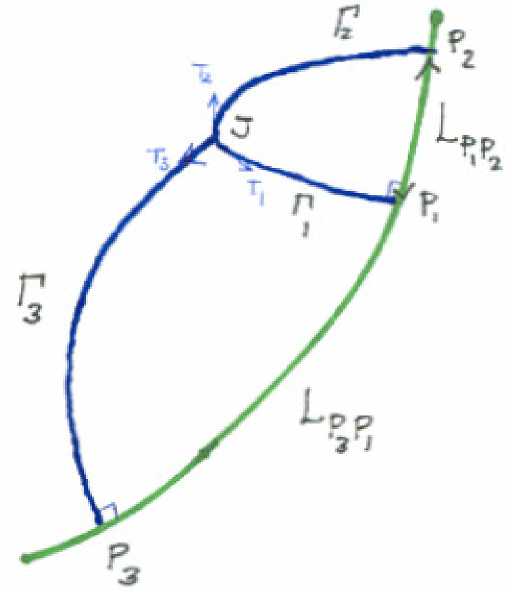


(S2 fails for any $\mathcal{O} \in \partial D$)

condition : $k_\Gamma \leq \pi/2$

$\Longrightarrow \exists\, \mathcal{O} \in \partial D$ satisfying (S2)

GEOMETRIC CONSTRAINTS



(A) Gauss–Bonnet

$$k_{\Gamma_1} + |k_{\Gamma_2}| = \frac{2\pi}{3} - L_{P_1P_2}$$

$$k_{\Gamma_3} - k_{\Gamma_1} = \frac{2\pi}{3} - L_{P_3P_1}$$

$$k_{\Gamma_3} + |k_{\Gamma_2}| = \frac{4\pi}{3} - L_{P_2P_3}$$

Consequences:

(i) For each t, either $k_{\Gamma_1} < \pi/3$ or $|k_{\Gamma_2}| < \pi/3$ (so $\mathcal{O}$ is always defined).

(ii) $\dot{k}_{\Gamma_j} = k(P_j) - \Delta_{1J}$: $L_{P_1P_2}\downarrow$, $L_{P_3P_2}\downarrow$

(iii) $k_{\Gamma_1} < \frac{\pi}{3} \Rightarrow |k_{\Gamma_2}| > \frac{\pi}{3} \Rightarrow k_{\Gamma_3} < \pi$ ($\mathcal{O}'$ is defined in this case)

(B) $A(t)$ = area enclosed by Γ_1, Γ_2: $\dot{A} = -(k_{\Gamma_1} + |k_{\Gamma_2}|) < 0$

Consequence: $L_{P_1P_2}(0) < \frac{\pi}{6} \Rightarrow L_{P_1P_2}(t) < \frac{\pi}{6} \Rightarrow \dot{A} = -\frac{2\pi}{3} + L_{P_1P_2} < -\frac{\pi}{2}$

$$\Rightarrow A(t) < A(0) - \frac{\pi}{2}t \Rightarrow T_{MAX} < \frac{2}{\pi}A(0)$$

(UPPER BOUND on time until junction hits the boundary).

CONTROL OF $\|k\|_{L^2}$

$$\frac{d}{dt}\int_\Gamma k^2 ds = \int_\Gamma k^4 ds - 2\int_\Gamma k_s^2 ds + 2k^2(P) - 2k\Lambda\Big|_J + k^2 v\Big|_J$$

all terms but Λ controlled via Sobolev inequalities (cp [Mantegazza]
Adding over branches this term disappears:

$$\frac{d}{dt}\int_{\mathbb{T}} k^2 ds \le C\left(\int_{\mathbb{T}} k^2 ds\right)^3$$

<u>Lemma</u> $\exists\, \epsilon_0 = \epsilon_0(\Delta)$ s.t.

$$\int_{\mathbb{T}} k^2 ds\Big|_{t=0} < \epsilon_0 \Rightarrow \int_{\mathbb{T}_t} k^2 ds < \left(\frac{\pi}{2}\right)^2 \quad \forall\, t < T_{max}$$

(in part. $\int_{\Gamma_3(t)} k^2 ds < \left(\frac{\pi}{2}\right)^2$).

<u>Cor.</u> Since $\dot{L_3} = v_3|_J - \int_{\Gamma_3^R} k_3^2 ds < 0$, we have:

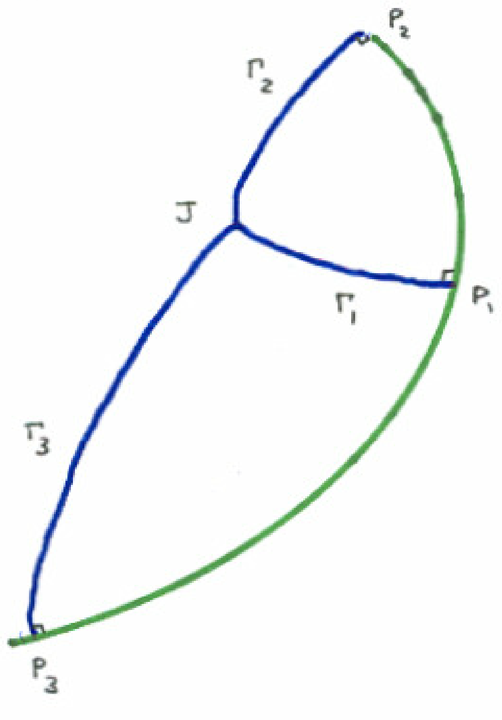
$$\int_{\Gamma_3(t)} k_3 ds \le L_3(t)^{1/2}\left(\int_{\Gamma_3} k_3^2 ds\right)^{1/2} \le \frac{\pi}{2} \quad \text{if } L_3(0) < 1.$$

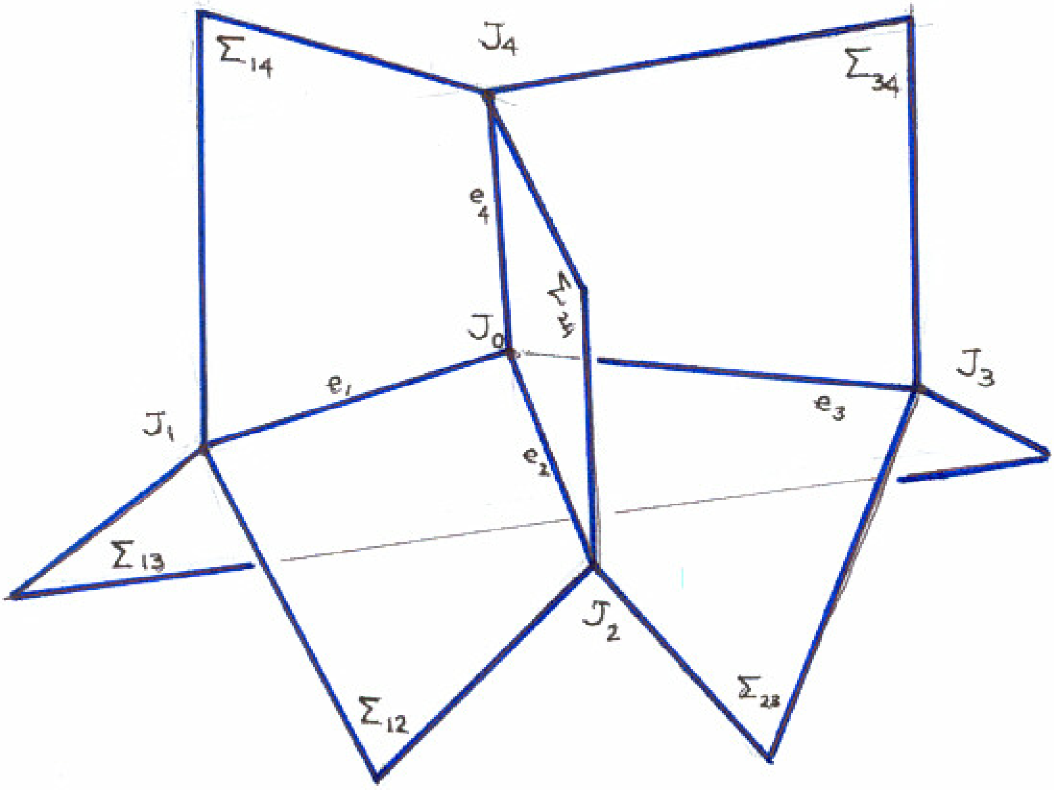
(so θ' is well-defined).

<u>Rk.</u> In general one cannot control $\left(\int_{\Gamma_3} k^2\right)^{\cdot}$ by $\int_{\Gamma_3} k^2$ alone.

THEOREM

Assume $k_1 > 0$, $k_2 < 0$, $k_3 > 0$ at $t = 0$.
Then these signs persist throughout the evolution.
If the solution is defined in $[0, T_{max})$ $(T_{max} < \infty)$, $\max_{\Gamma_1 \cup \Gamma_2} |k|$ is bounded above as long as $L_{P_1 P_2}$ is bounded below. If, in addition, the total curvature k_{Γ_3} of Γ_3 is smaller than π in $[0, T_{max})$, $\sup_{\pi} |k|$ is bounded above, and the evolution continues until (at $t = T_{max}$) the triple junction meets the boundary.





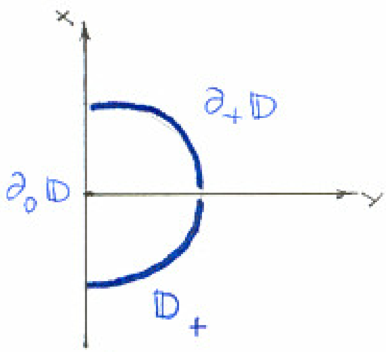
QUADRAJUNCTION IN $\mathbb{R}^3$

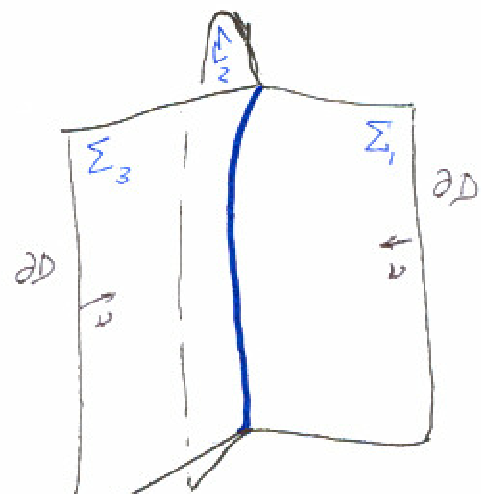
six surfaces $\Sigma_{12}, \ldots, \Sigma_{34}$ meeting at J_0 (tetrahedral angle)

4 edges e_1, e_2, e_3, e_4 w/ 120° dihedral angles

MOTION OF 4-JUNCTIONS IN $\mathbb{R}^3$

Model problem





$$u^j : \mathbb{D}_+ \times [0,T) \to D, \qquad j = 1,2,3$$

PDE:

$$u^j_t = g^{ab}(\nabla u^j) \frac{\partial^2 u^j}{\partial x^a \partial x^b}, \qquad g_{ab}(\nabla u^j) = \langle \frac{\partial u^j}{\partial x^a}, \frac{\partial u^j}{\partial x^b} \rangle$$

BC at $\partial_+ \mathbb{D}$:

$$b(u^j) = 0, \quad \langle N^j, \nu(u^j) \rangle = 0.$$

BC at $\partial_0 \mathbb{D}$:

$$u^1(x,0,t) = u^2(x,0,t) = u^3(x,0,t)$$

$$\langle N^1, N^2 \rangle = -\frac{1}{2} = \langle N^2, N^3 \rangle.$$

IC:

$$u^j|_{t=0} = u^j_0 : \mathbb{D}_+ \to D.$$

LOCAL EXISTENCE SCHEME

Linearization at u_0

LPDE:

$$u_t^j - g^{ab}(\nabla u_0^j)\frac{\partial^2 u^j}{\partial x^a \partial x^b} = f^j(\bar{u})$$

LBC at $\partial_0\mathbb{D}$:

$$u^1 = u^2 = u^3$$
$$\langle u_x^1, N_0^2\rangle - \langle N_0^1, u_x^2\rangle = \Phi(\bar{u})$$
$$\langle u_x^2, N_0^3\rangle - \langle N_0^2, u_x^3\rangle = \Psi(\bar{u})$$

LBC at $\partial_+\mathbb{D}$:

$$\langle \nu(u_0^j), u^j\rangle = \Omega(\bar{u})$$
$$\langle \nu(u_0^j), u_r^j\rangle - A^{bdry}(u^{jT}, N_0^j) = \chi(\bar{u})$$

IC:

$$u^j|_{t=0} = u_0^j$$

Goal:

(i) (LPDE/LBC) uniquely solvable in $C^{2+\alpha,1+\alpha/2}$, for $(u_0, f, \Phi, \Psi, \Omega, \chi) \in E$

(ii) Let $V = \{u^j \in C^{2+\alpha,1+\alpha/2}; u^j|_{t=0} = u_0^j, (u^j) \text{ sat. the compatibility conditions}\}$.
The composition:

$$\begin{aligned} V &\to E \to V \\ \bar{u} &\mapsto (u_0, f, \Phi, \Psi, \Omega, \chi) \mapsto u \end{aligned}$$

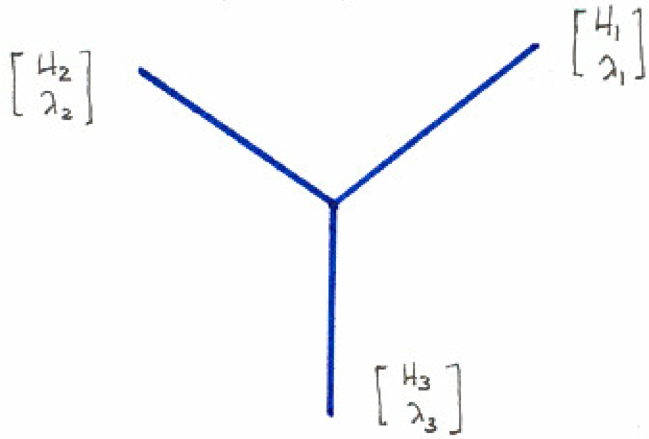
is a contraction.

COMPATIBILITY CONDITIONS (t=0)

$$u_t^j = \lambda_0 e_0 + \lambda_j e_j + H_j N^j$$

(I) At an *edge*:

$$\sum_i H_i = \sum_i \lambda_i = 0$$



$$e_1(H_1) + H_1\lambda_1 = e_2(H_2) + H_2\lambda_2 = e_3(H_3) + H_3\lambda_3$$

(II) At the boundary:

$$\langle \lambda_0 e_0 + \lambda_1 e_1, \nu\rangle|_{\partial_+\mathrm{D}} = 0$$

$$\frac{\partial H}{\partial n} = HA^{bdry}(N, N)$$

COMPLEMENTARITY

I. At $\partial_0 \mathbb{D}_+ = \{y = 0\}$

$$u^j(x,y,t) \longrightarrow \tilde{u}^j(\xi, y, p) \text{ (Laplace in t, Fourier in x)}$$

$$p\tilde{u}^j + g_j^{11}\xi^2\tilde{u}^j + 2ig_j^{12}\xi\frac{\partial\tilde{u}^j}{\partial y} - g_j^{22}\frac{\partial^2 u^j}{\partial y^2} = 0$$

$$\tilde{u}^j = c^j(\xi,p)e^{-\alpha_j y}$$

(All α_j are equal if the u_0^j are conformal parametrizations)

$$c^1(\xi,p) = c^2(\xi,p) = c^3(\xi,p) := c(\xi,p)$$
$$\langle c, \alpha u_{0x} \wedge (N_0^1 + N_0^2)\rangle = \tilde{\Phi}$$
$$\langle c, \alpha u_{0x} \wedge (N_0^2 + N_0^3)\rangle = \tilde{\Psi}$$

These conditions are l.i., but there is still one missing!

Consider the extra condition:

$$(NL) \quad \langle u_y^1, u_x^1\rangle - \langle u_y^2, u_x^2\rangle - \langle u_y^3, u_x^3\rangle|_{\partial_0\mathbb{D}} = 0$$

This leads to:

$$\langle c, \alpha u_{0x}\rangle = \tilde{\Lambda},$$

the missing condition for c.

II. At the boundary

As additional condition we take: $(NL) \quad \langle u_x, u_y\rangle = 0$, leading to:

$$\langle c, i\xi u_{0y} - \alpha u_{0x}\rangle = \tilde{\Xi}$$

Lemma:

1. The extended (LBC) satisfy the complementarity condition on $\partial\mathbb{D}_+$.
2. With conformally parametrized u_0^j satisfying the compatibility conditions, the model problem has a unique short-time solution.

LOCAL EXISTENCE FOR QUADRAJUNCTIONS

Surfaces Σ^I, $I \in \{(12),(13),(14),(23),(24),(34)\}$ parametrized by:

$$u^I : Q \times [0,T) \to D \subset \mathbb{R}^3$$

Local existence theorem. The system:

$$u_t^I = g_I^{ab} \frac{\partial^2 u^I}{\partial x^a \partial x^b}$$

with Neumann boundary conditions, Plateau conditions at the edges, and conformally parametrized initial surfaces satisfying the compatibility conditions has a unique short-time solution:

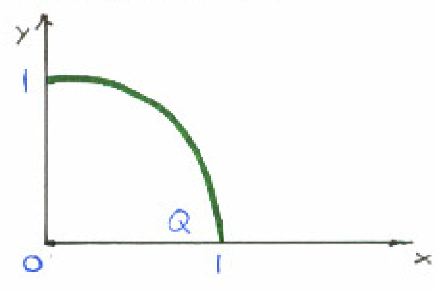
$$u^I \in C^{2+\alpha, 1+\alpha/2}(Q \times [0,T); D)$$

(satisfying also the extended boundary conditions (NL).)

Regularity:(i) The maps $u^I(\cdot,t)$ extend $C^{1+\alpha}$ to $\bar{Q} - \{(0,0),(1,0),(0,1)\}$ and C^β to $\bar{Q}$, for any $0 < \beta < 1/2$.

(ii) The unit normals N^I extend continuously to the closure $\bar{Q}$.

(Parabolic system in a domain with corners.)



Condition count:

At each edge (e_1, e_2 at $\{x = 0\}$, e_3, e_4 at $\{y = 0\}$):

Two independent incidence conditions (6 scalar conditions)+ two angle conditions + 1 extra: 9 scalar conditions.

This yields 18 scalar conditions at $\{x = 0\}$ and 18 at $\{y = 0\}$, matching the 18 scalar components for u^I.

Still open: Geometric uniqueness of the evolution (up to reparametrization), continuation criterion (ideally in terms of boundedness of $|A|^2$, the second fundamental form)