Coarse Equivalence and $\mathbb{Q}$

E. Weir

Abstract

This presentation examines metrics on $\mathbb{Q}$ and some of its subgroups with the intent of classifying groups up to coarse equivalence. Properness of these metrics is accompanied by estimates for polynomial or exponential growth. It is shown that certain subgroups of $\mathbb{Q}$ can be used in the pursuit of determining whether $\bigoplus_\infty \mathbb{Z}_2$ is coarsely equivalent to $\bigoplus_\infty \mathbb{Z}_3$.

Definition Two groups X and Y with metrics d and d' and maps $f : X \mapsto Y$ and $g : Y \mapsto X$ are said to be ***coarsely equivalent*** if they satisfy the following properties:

1. Their compositions are c-close to the identity
   (a) $\exists c$ such that $\forall x \in X$, $d\left(g \circ f\left(x\right), x\right) \leq c$.
   (b) $\exists c$ such that $\forall y \in Y$, $d\left(f \circ g\left(y\right), y\right) \leq c$.

2. f and g are ρ-bornologous
   (a) $\exists \rho : \mathbb{R}^{+} \mapsto \mathbb{R}^{+}$ increasing such that $\forall x, x' \in X$, $d\left(f\left(x\right), f\left(x'\right)\right) \leq \rho\left(d\left(x, x'\right)\right)$
   (b) $\exists \rho' : \mathbb{R}^{+} \mapsto \mathbb{R}^{+}$ increasing such that $\forall y, y' \in Y$, $d\left(g\left(y\right), g\left(y'\right)\right) \leq \rho'\left(d\left(y, y'\right)\right)$

Definition A left-invariant metric is called ***proper*** if $\forall R > 0$, $|B\,(0, R)|$ is finite.

- Example: Euclidean metric on $\mathbb{R}$
- Example: Euclidean metric on $\mathbb{Z}$

Description A ***filtration*** on a group G is a sequence of subgroups like the following:

$$\mathcal{L} : \{e \subset G_1 \subset G_2 \subset G_3 \subset ... \subset G\}$$

where $G_0 = e$, and $G_i \subset G_{i+1}$.

- Example: A filtration on $\bigoplus_\infty \mathbb{Z}_2$ is

$$\mathcal{L} : \{0 \subset \mathbb{Z}_2 \subset \mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset ...\}$$

The $d\mathcal{L}$ metric A filtration on a group can be used to define a metric:

$$d\mathcal{L}\,(x, y) = \min\{i \mid |x - y| \in G_i\}$$

- Example for $\bigoplus_\infty \mathbb{Z}_2$:

$$\begin{aligned} d\mathcal{L}\left(01011\overline{0}, 11\overline{0}\right) &= \min\left\{i \mid \left|01011\overline{0} - 11\overline{0}\right| \in G_i\right\} \\ &= \min\left\{i \mid 10011\overline{0} \in G_i\right\} = 5 \end{aligned}$$

Define a filtration on $\mathbb{Q}_2/\mathbb{Z}$ where $k \in \mathbb{N}$ as follows:

$$\mathcal{L} : 0 \subset \left\langle \frac{1}{2} \right\rangle \subset \left\langle \frac{1}{4} \right\rangle \subset \ldots \subset \left\langle \frac{1}{2^k} \right\rangle$$

Use this filtration to create the metric:

$$d\mathcal{L}\left(x, y\right) = \min\left\{i \mid |x - y| \in G_i\right\}.$$

Proving $d\mathcal{L}\left(x,y\right) = \min\left\{i \mid |x-y| \in G_i\right\}$ on $\mathbb{Q}_2/\mathbb{Z}$ is a metric:

- Positive definiteness, symmetry, and left-invariance
  - Follows from definition of metric
- Triangle Inequality
  - Maximum argument
- Properness
  - Any ball $B\left(0,R\right)$ contains at most 2^R-1 elements

Proving $X = \bigoplus_{\infty} \mathbb{Z}_2 \simeq \mathbb{Q}_2/\mathbb{Z} = Y$

- Use the $d\mathcal{L}$ metric for each

- Define the mappings as follows:

$$f : X \mapsto Y$$

$\forall x = \{x_1, x_2, \ldots, x_k\} \in X$, $f(x) = \frac{m}{2^k}$ where m is the binary value of x and k is its length.

$$g : Y \mapsto X$$

$\forall y = \frac{m}{2^k} \in Y$, $g(y)$ is equal to a binary string with value m and length k.

- This creates a bijection
  - $f \circ g = g \circ f = id$

- $d\left(f\left(x\right), f\left(x'\right)\right) = d\left(x, x'\right)$ and $d\left(g\left(y\right), g\left(y'\right)\right) = d\left(y, y'\right)$
  - Metrics are essentially the same

- This means that $X \simeq Y$

- A similar argument works for $\bigoplus_{\infty} \mathbb{Z}_3 \simeq \mathbb{Q}_3/\mathbb{Z}$
  - Since coarse equivalence is an equivalence relation,

$$\mathbb{Q}_2/\mathbb{Z} \simeq \mathbb{Q}_3/\mathbb{Z} \Leftrightarrow \bigoplus_{\infty} \mathbb{Z}_2 \simeq \bigoplus_{\infty} \mathbb{Z}_3$$

  - So $\mathbb{Q}_2/\mathbb{Z}$ can be used for the main problem

Define a filtration on $\mathbb{Q}_2$ as was defined on $\mathbb{Q}_2/\mathbb{Z}$. The metric on $\mathbb{Q}_2$ is defined as $d(x, y) = |x - y| + d\mathcal{L}(x, y)$.

Proving $d(x, y) = |x - y| + d\mathcal{L}(x, y)$ is a metric:

- Positive definiteness, symmetry, and left-invariance
  - Follows from definition of metric
- Triangle Inequality
  - Similar to $\mathbb{Q}_2/\mathbb{Z}$
- Properness
  - Any ball $B(0, R)$ contains at most $2 * \left(2^R * R\right)$ elements

Proving $X = \bigoplus_{\infty} \mathbb{Z}_2 \oplus \mathbb{Z} \simeq \mathbb{Q}_2 = Y$

- Use the metric previously defined on $\mathbb{Q}_2$ for both

- Define the mappings as follows:

$$f : X \mapsto Y$$

$\forall x = \{x_z, x_1, \ldots, x_k\} \in X$, where x_z is the element of $\mathbb{Z}$, and x_k is the last nonzero digit of $\bigoplus_{\infty} \mathbb{Z}_2$, $f(x) = x_k + \frac{p}{2^k}$, where p is the binary value of the string and k is its length.

$$g : Y \mapsto X$$

$\forall y = m + \frac{p}{2^k} \in Y$, where m is an integer and $0 \leq \frac{p}{2^k} < 1$, $g(y) = m \oplus b$, where b is a binary string of length k with value p.

- Similar arguments as for $\bigoplus_{\infty} \mathbb{Z}_2$ and $\mathbb{Q}_2/\mathbb{Z}$ show that these mappings make $X \simeq Y$.

Define a filtration on $\mathbb{Q}$ where $k \in \mathbb{N}$ as follows:

$$\mathcal{L} : 0 \subset \left\langle \frac{1}{2} \right\rangle \subset \left\langle \frac{1}{2}, \frac{1}{3} \right\rangle \subset \ldots \subset \left\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{k} \right\rangle$$

with each subgroup labeled like this:

$$G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_k$$

Use this filtration to create a metric:

$$d\mathcal{L}\left(x, y\right) = \min\left\{i \mid |x - y| \in G_i\right\}.$$

Define $d(x,y) = |x-y| + d\mathcal{L}(x,y)$, and prove it is a metric:

- Positive definiteness, symmetry, and left-invariance
  - Follows from definition of the metric
- Triangle Inequality
  - Similar to $\mathbb{Q}_2/\mathbb{Z}$
- Properness
  - Any ball $B(0,R)$ contains at most $R^3 - R^2 + 1$ elements

Properness of $\mathbb{Q}$:

- The following are added for each increase of 1 in the value of R:
    - Fractions with denominator R: $< R$
    - Fractions with denominator $R-1$: $< R+1$
    - $\vdots$
    - Fractions with denominator 1 (integers): 1
- So the number of elements added between R and $R-1$ is $< \sum_{i=1}^{R} i = \frac{(R)(R-1)}{2}$

- Multiply by 2 to account for negative fractions and R to account for fractions added in previous balls
  - $|B(0,R)| < (2R)\left[\frac{(R)(R-1)}{2}\right] = R^3 - R^2$
- Add one to account for 0
  - $|B(0,R)| < R^3 - R^2 + 1$

Thanks to Dr. N. Brodskiy and Dr. J. Conant for advising me in my research, and to my fellow GT group members Kristine Buddemeyer, Hassane Kone, and Tamara Dietrich-Muller for their contributions and explanations.