Coarse Equivalence in Cyclic Groups

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Abstract

This paper summarizes the results from the Summer 2007 Mathematics REU at the University of Tennessee. Knoxville.

1 Coarse Equivalence

Definition 1.1 Two groups X and Y with metrics d and d' and maps $f : X \mapsto Y$ and $g : Y \mapsto X$ are said to be ***coarsely equivalent*** if they satisfy the following properties:

1. Their compositions are c-close to the identity
   (a) $\exists c$ such that $\forall x \in X$, $d\left(g \circ f\left(x\right), x\right) \leq c$.
   (b) $\exists c$ such that $\forall y \in Y$, $d\left(f \circ g\left(y\right), y\right) \leq c$.
2. f and g are ρ-bornologous
   (a) $\exists \rho : \mathbb{R}^+ \mapsto \mathbb{R}^+$ increasing such that $\forall x, x' \in X$, $d\left(f\left(x\right), f\left(x'\right)\right) \leq \rho\left(d\left(x, x'\right)\right)$
   (b) $\exists \rho' : \mathbb{R}^+ \mapsto \mathbb{R}^+$ increasing such that $\forall y, y' \in Y$, $d\left(g\left(y\right), g\left(y'\right)\right) \leq \rho'\left(d\left(y, y'\right)\right)$

Definition 1.2 A left-invariant metric is called ***proper*** if $\forall R > 0$, $\left|B\left(0, R\right)\right|$ is finite.

Theorem 1.3 Coarse equivalence is an equivalence relation.

Note: We will use $\simeq$ to denote coarse equivalence.

Proof

1. *Coarse Equivalence is reflexive.* For all metric spaces $(X, d), \exists\, f : X \to X, f = id_x$ where f is ρ-bornologous, with $\rho = id$. It is clear that $f \circ f$ is c-close to the identity $\forall c > 0$.

2. *Coarse Equivalence is symmetric.* $X \simeq Y \Rightarrow \exists\, f : X \to Y, g : Y \to X$ such that f, g are ρ-bornologous and $\exists\, c > 0$ where $f \circ g, g \circ f$ are c-close to the identity map. We can simply re-label f and g to complete the proof.

3. *Coarse Equivalence is transitive.* $X \simeq Y, Y \simeq Z \Rightarrow \exists\, f_1 : X \to Y, g_1 : Y \to X, f_2 : Y \to Z, g_2 : Z \to Y$ where f_1, g_1, f_2, g_2 are ρ-bornologous for some $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and $\exists\, c > 0$ such that $d(g_1 \circ f_1(x), x) < c$, $d(f_1 \circ g_1(y), y) < c$, $d(g_2 \circ f_2(y), y) < c$ and $d(f_2 \circ g_2(z), z) < c$.

Claim 1: Composition of bornologous maps is bornologous. $f_1 : X \to Y$, $f_2 : Y \to X$ bornologous $\Rightarrow \forall x, x' \in X$ and $y, y' \in Y\, \exists\, \rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d(f_1(x), f_1(x')) \leq \rho_1(d(x, x'))$ and $d(f_2(x), f_2(x')) \leq \rho_2(d(x, x'))$. Thus, $d(f_2 \circ f_1(x), f_2 \circ f_1(x')) \leq \rho_2(d(f_1(x), f_1(x')))$. Assuming ρ_1, ρ_2 are increasing, we have $\rho_2(d(f_1(x), f_1(x'))) \leq \rho_2(\rho_1(d(x, x'))) = \rho_2 \circ \rho_1(d(x, x'))$. Therefore, $d(f_2 \circ f_1(x), f_2 \circ f_1(x')) \leq \rho_2 \circ \rho_1(d(x, x'))$, i.e $f_2 \circ f_1$ is $\rho_2 \circ \rho_1$-bornologous.

For c-closeness, it suffices to show that $g_1 \circ g_2 \circ f_2 \circ f_1 \approx_c id_x$ for some $c > 0$. $Y \simeq Z \Rightarrow d(g_2 \circ f_2(y), y) \leq c\,, \forall y \in Y$. Let $y = f_1(x)$. Then $d(g_2 \circ f_2 \circ f_1(x), f_1(x)) \leq c\,, \forall x \in X$. g_1 is ρ-bornologous $\Rightarrow d(g_1 \circ g_2 \circ f_2 \circ f_1(x), g_1 \circ f_1(x)) \leq \rho(d(g_2 \circ f_2 \circ f_1(x), f_1(x)) \leq \rho(c)$. By the triangle inequality, $d(g_1 \circ g_2 \circ f_2 \circ f_1(x), x) \leq d(g_1 \circ g_2 \circ f_2 \circ f_1(x), g_1 \circ f_1(x)) + d(g_1 \circ f_1(x), x) \leq \rho(c) + c$. Let $c' = \rho(c) + c$ and $g_1 \circ g_2 \circ f_2 \circ f_1(x) \approx_{c'} id_x$.

Theorem 1.4 Let $\mathcal{L}_G = G_0 \subset G_1 \subset G_2 \ldots \subset G$ where $G_0 = 1$ and $\mathcal{L}_H = H_0 \subset H_1 \subset H_2 \ldots \subset G$ where $H_0 = 1$ be two different filtrations of the group $(G, *)$ by finite subgroups. Then $(G, d_{\mathcal{L}_G})$ is bijectively coarse equivalent to $(G, d_{\mathcal{L}_H})$.

Lemma 1.5 Let $f : (G, d_{\mathcal{L}_G}) \to (G, d_{\mathcal{L}_H})$ where $f(g) = g, \forall g \in G$. For all $x, y \in G, d_{\mathcal{L}_G}(x, y) = min\{i |\, x^{-1}y \in G_i\}$ and $d_{\mathcal{L}_H}(x, y) = min\{i |\, x^{-1}y \in H_i\}$. Let $d_{\mathcal{L}_G}(x, y) = n$. For all $n, \exists\; i$ such that $G_n \subset H_i$. Let $\rho(n) = min\{i | G_n \subset H_i\}$.Then $\rho : \mathbb{N} \to \mathbb{N}$ is an increasing function.

Proof of Lemma: Since $\mathcal{L}_G$ is a filtration, $\forall n, k \in \mathbb{N}$, $G_n \subset G_{(n+k)}$. By the definition of ρ, $G_n \subset H_{\rho(n)}$. Likewise, $G_{n+k} \subset H_{\rho(n+k)}$.

Claim 1: $\forall n, k \in \mathbb{N}$, $H_{\rho(n)} \subset H_{\rho(n+k)}$.

Proof of Claim: Suppose this claim was not true. Then we would have $H_{\rho(n+k)}$ strictly contained in $H_{\rho(n)}$. So, $G_n \subset G_{n+k} \subset H_{\rho(n+k)} \subset H_{\rho(n)}$ where

$$H_{\rho(n+k)} \neq H_{\rho(n)}$$

But, $G_n \subset H_{\rho(n+k)} \subset H_{\rho(n)}$ where $H_{\rho(n+k)} \neq H_{\rho(n)}$ is a contradiction of the definition that $H_{\rho(n)}$ is the smallest H_i such that $G_n \subset H_i$.

Thus, we have $\forall n, k \in \mathbb{N}$, $H_{\rho(n)} \subset H_{\rho(n+k)}$. So, because $\mathcal{L}_H$ is a filtration, $\rho(n) \leq \rho(n + k)$. Therefore, ρ is an increasing function.

Proof of Theorem 1.4: Again, let $f : (G, d_{\mathcal{L}_G}) \to (G, d_{\mathcal{L}_H})$ where $f(g) = g, \forall g \in G$. For all $x, y \in G, d_{\mathcal{L}_G}(x, y) = min\{i|\, x^{-1}y \in G_i\}$ and $d_{\mathcal{L}_H}(x, y) = min\{i|\, x^{-1}y \in H_i\}$. Let $d_{\mathcal{L}_G}(x, y) = n$. For all $n, \exists\, i$ such that $G_n \subset H_i$. Let $\rho(n) = min\{i|G_n \subset H_i\}$. By **Lemma 1.3**, we have $\rho(n)$ is an increasing function.

Claim 2: $d_{\mathcal{L}_H}(x, y) = min\{i|x^{-1}y \in H_i\} \leq min\{i|G_n \subset H_i\} = \rho(n) = \rho(d_{\mathcal{L}_G}(x, y))$.

Proof of Claim: Let $k = d_{\mathcal{L}_H}(x, y)$. So, $k = d_{\mathcal{L}_H}(x, y) \leq \rho(n) \Rightarrow H_k \subset H_{\rho(n)}$. If this were not true, then $H_{\rho(n)} \subset H_k$ with $H_{\rho(n)} \neq H_k$. But this would contradict the assumption that H_k is the smallest H_i containing $x^{-1}y$.

So, $\forall x, y \in G, d_{\mathcal{L}_H}(x, y) \leq \rho(d_{\mathcal{L}_G}(x, y))$. Here, we have shown that f is ρ-bornologous with $\rho(n) = min\{i|G_n \subset H_i\}$. f is the identity function on G, therefore a bijection with $f = f^{-1}$. By the same reasoning as above, f^{-1} is ρ'-bornologous with $\rho'(n) = min\{i|H_n \subset G_i\}$. Thus, $(G, d_{\mathcal{L}_G})$ is bijectively coarse equivalent to $(G, d_{\mathcal{L}_H})$.

2 Classifying Basic Groups Up To Coarse Equivalence

Example 2.1 $X = \bigoplus_{\infty} Z_2 \overset{coarse}{\simeq} \bigoplus_{i=2}^{\infty} (\mathbb{Z}_2)_i = Y$

Proof: $\bigoplus_{i=2}^{\infty} (\mathbb{Z}_2)_i$ is the same as $0 \oplus \bigoplus_{i=2}^{\infty} (\mathbb{Z}_2)_i$. So using the d$\mathcal{L}$ metric, $\forall x \in X$ where $x = \{x_1, x_2, x_3, \ldots, x_n\}$ and $\forall y \in Y$ where $y = \{0, y_2, y_3, \ldots, y_n\}$:

$$\begin{aligned} f : X &\mapsto Y \\ f(x_1, x_2, x_3, \ldots, x_n) &= \{0, x_2, x_3, \ldots, x_n\} \\ g : Y &\mapsto X \\ g(y) &= y \end{aligned}$$

By this mapping, $f \circ g = id_Y$ and $g \circ f \overset{1}{\approx} id_X$, since any value input will only differ in the first digit. $d(f(x), f(x')) \leq d(x, x')$, since if the point of difference is at $x_2$or above, the distance is the same. If the point of difference is at x_1, the distance between the two points is zero. $d(g(y), g(y')) = d(y, y')$, since g is the identity. So the conditions of c-closeness (where c=1) and ρ-bornologousness are satisfied, thus $X \overset{coarse}{\simeq} Y$.

Example 2.2 $\mathbb{Z}_3 \oplus \bigoplus_{i=2}^{\infty} (\mathbb{Z}_2)_i \overset{coarse}{\simeq} \bigoplus_{i=2}^{\infty} (\mathbb{Z}_2)_i$

Proof: Same as Example 2.1, with the first $\mathbb{Z}_2$replaced with a $\mathbb{Z}_3$.

Example 2.3 $X = \mathbb{Z}_a \bigoplus_{\infty} \mathbb{Z}_c \overset{coarse}{\simeq} \mathbb{Z}_b \bigoplus_{\infty} \mathbb{Z}_c = Y$

Proof: Using the d$\mathcal{L}$ metric, $\forall x \in X$ where $x = \{x_1, x_2, x_3, \ldots, x_n\}$ and $\forall y \in Y$ where $y = \{y_1, y_2, y_3, \ldots, y_n\}$:

$$\begin{aligned} f : X &\mapsto Y \\ f(x_1, x_2, x_3, \ldots, x_n) &= \{0, x_2, x_3, \ldots, x_n\} \\ g : Y &\mapsto X \\ g(y_1, y_2, y_3, \ldots, y_n) &= \{0, y_2, y_3, \ldots, y_n\} \end{aligned}$$

By this mapping, $f \circ g \overset{1}{\approx} id_Y$ and $g \circ f \overset{1}{\approx} id_X$, since the only point of difference induced by the compositions is on the first digit. $d\left(f\left(x\right), f\left(x'\right)\right) \leq d\left(x, x'\right)$, since no new points of difference are introduced by f, and likewise for g. Thus $X \overset{coarse}{\simeq} Y$.

Example 2.4 $X = \bigoplus_{i=1}^{n} \left(\mathbb{Z}_a\right)_i \bigoplus G \overset{coarse}{\simeq} \bigoplus_{i=1}^{n} \left(\mathbb{Z}_b\right)_i \bigoplus G = Y$

Proof: G is an infinite group, which can be represented in the direct sum as $g \in G$. So using g as the last element in the $d\mathcal{L}$ filtration, we have the following:

$$\begin{aligned} f : X &\mapsto Y \\ f(x_1, x_2, \ldots, x_n, g_a) &= \{0, 0, 0, \ldots, g_a\} \\ g : Y &\mapsto X \\ g(y_1, y_2, \ldots, y_n, g_b) &= \{0, 0, 0, \ldots, g_b\} \end{aligned}$$

Both $f \circ g$ and $g \circ f$ are n-close to the identity, and $d\left(f\left(x\right), f\left(x'\right)\right) \leq d\left(x, x'\right)$, since any points of difference other than g_a are made the same. A similar argument holds for g. Thus $X \overset{coarse}{\simeq} Y$.

Example 2.5 $X = H_1 \bigoplus G \overset{coarse}{\simeq} H_2 \bigoplus G = Y$

Proof: Let H_1, H_2be finite groups. For some finite n, $h \in H_1$can be represented as $\{h_1, h_2, \ldots, h_n\}$, and similarly for $g \in H_2$. By an argument similar to Example 2.4, the two groups are then coarsely equivalent.

3 Metrics

Lemma 3.1 If d is a proper left invariant metric, then f is increasing, $f(0) = 0$ and f is concave $\Rightarrow f\left(d\right)$ is a proper left invariant metric.

Proof For, we just need to check the triangle inequality: $f(d(x, z)) \leq f(d(x, y) + d(y, z)) \leq f(d(x, y)) + f(d(y, z))$.

Lemma 3.2 For any set A and a metric space (X, d), we can define a metric on A by defining a injective function $f : A \rightarrow X$ by $d'(x, y) = d(f(x), f(y))$ where $x, y \in A$. d' is proper left invariant $\Leftrightarrow f$ is linear.

Proof Since f is injective then $x \neq y$ and $d'(x,y) = 0$ is remote. We just need to check the triangle inequality again: $d'(x,z) = d(f(x), f(z)) \leq d(f(x), f(y)) + d(f(y), f(z)) = d'(x,y) + d'(y,z)$

f linear $\Leftrightarrow f(x-y) = f(x) - f(y) \Leftrightarrow d'(x,y) = d(f(x), f(y)) = d(0, f(y) - f(x)) = d(0, f(y-x)) = d'(0, y-x)$

3.1 Metric on $\mathbb{Q}_2/\mathbb{Z}$

Create a filtration on $\mathbb{Q}_2/\mathbb{Z}$ as follows:$\mathcal{L} : 0 \subset \left\langle \frac{1}{2} \right\rangle \subset \left\langle \frac{1}{4} \right\rangle \subset \ldots \subset \left\langle \frac{1}{2^k} \right\rangle$, where $k \in \mathbb{N}$. Define distance as $dL(x,y) = \min \{ i \, || x - y| \in G_i \}$. Positive definiteness and symmetry follow from the definition of distance, leaving the triangle inequality to show that this is a metric.

Proof of triangle inequality: $\forall \frac{m}{2^j}, \frac{n}{2^k}, \frac{p}{2^l} \in \mathbb{Q}_2/\mathbb{Z}$, show that:

$$d\left(\frac{m}{2^j}, \frac{p}{2^l}\right) \leq d\left(\frac{m}{2^j}, \frac{n}{2^k}\right) + d\left(\frac{n}{2^k}, \frac{p}{2^l}\right) \tag{1}$$

This means that it must be shown that $\max \{j, l\} \leq \max \{j, k\} + \max \{k, l\}$.

Assuming $j \leq k \leq l$, this reduces to showing that $l \leq k + l$, which is true.

Assuming $j \leq l \leq k$, this reduces to showing that $l \leq k + k$, which is true, since $k \geq l$.

Assuming $k \leq l \leq j$, this reduces to showing that $j \leq j + l$, which is true.

Assuming $k \leq j \leq l$, this reduces to showing that $l \leq j + l$, which is true.

Assuming $l \leq j \leq k$, this reduces to showing that $j \leq k + k$, which is true, since $k \geq j$.

Assuming $l \leq k \leq j$, this reduces to showing that $j \leq j + k$, which is true.

So the triangle inequality holds.

It must also be shown that the metric is proper and left-invariant. Left invariance is obvious since the distance is defined by subtracting two points. Properness is easily shown, since each filtration can have only $2_k - 1$ elements of the form $\frac{m}{2^k}$ before it cycles back because of mod $\mathbb{Z}$.

3.2 Ultrametric on $\mathbb{Q}/\mathbb{Z}$

Create a filtration on $\mathbb{Q}/\mathbb{Z}$ as follows: $\mathcal{L} : 0 \subset \left\langle \frac{1}{2} \right\rangle \subset \left\langle \frac{1}{2}, \frac{1}{3} \right\rangle \subset \left\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \right\rangle \subset \ldots \subset \left\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n} \right\rangle$where $n \in \mathbb{N}$. Define distance as $dL(x,y) = \min \{ i \, || x - y| \in G_i \}$. One can make similar arguments as for the triangle inequality in **3.2**, noting that the difference of any x, x' who belong to G_a, G_b respectively where $a \leq b$ is itself an element of G_b. Using this, the same arguments for maxima apply. Also similarly, left-invariance is obvious, and properness comes from the mod $\mathbb{Z}$, since any filtration cannot have more than $n - 1$ elements of the form $\frac{m}{n}$.

3.3 Metric on $\mathbb{Q}$

Define the $d\mathcal{L}$ metric as in **3.3**, but instead of measuring distance strictly by that metric, set $d(x,y) = |x-y| + d\mathcal{L}$. Again, positive definiteness and symmetry are clearly properties of this metric, and the triangle inequality can be shown in a similar way to **3.3**, as can left invariance. The properness of the metric, however, is not obvious.

One can conjecture that since the metric on $\mathbb{Q}_2$ is proper such that$|B(0,R)| \leq \left(R * 2 * 2^R\right)$, $\mathbb{Q}$ should be similarly proper.

3.4 Metrics on $\mathbb{Z}$

Lemma 3.5.1 Any proper left invariant metric on $\mathbb{Z}$ comes from applying a function to the usual Euclidian metric of $\mathbb{Z}$.

Proof Let d be an arbitrary proper left invariant on $\mathbb{Z}$, then $d(x,y) = d(0,y-x)$ and $\exists f : \mathbb{Z} \to \mathbb{R}$ such that $d(0,s) = f(s)$. But $f(-s) = d(0,-s) = d(s,0) = d(0,s) = f(s)$, so $d(0,y-x) = f(|y-x|)$.

4 Maps

4.1 The Binary Mapping

One strategy used to try to find a mapping between $\bigoplus_\infty \mathbb{Z}_2$ and $\bigoplus_\infty \mathbb{Z}_3$was to use the idea of binary numbers in $\bigoplus_\infty \mathbb{Z}_2$ to create a map between the two groups. Instead of taking each position of $\bigoplus_\infty \mathbb{Z}_2$ on its own, pairs of ones and zeroes were used to map onto $\bigoplus_\infty \mathbb{Z}_3$. So for $x \in\bigoplus_\infty \mathbb{Z}_2$ where $x = \{x_1, x_2, x_3, x_4, \ldots, x_n, \ldots\}$, $x_1$and x_2 would be mapped onto one position in $\bigoplus_\infty \mathbb{Z}_3$, as would x_3 and x_4, and so forth. The following maps were considered:

$$
\begin{aligned}
f : \bigoplus\nolimits_\infty \mathbb{Z}_2 &\mapsto \bigoplus\nolimits_\infty \mathbb{Z}_3 \\
y_n = f\left(x_{2n}, x_{2n+1}\right) &= \begin{cases} x_{2n} + x_{2n+1} \end{cases} \\
g : \bigoplus\nolimits_\infty \mathbb{Z}_3 &\mapsto \bigoplus\nolimits_\infty \mathbb{Z}_2 \\
\{x_{2n}, x_{2n+1}\} = g\left(y_n\right) &= \begin{cases} \{0,0\} & if \quad y_n = 0 \\ \{0,1\} & if \quad y_n = 1 \\ \{1,1\} & if \quad y_n = 2 \end{cases}
\end{aligned}
$$

The primary issue with this mapping is that the compositions of these functions under the $d\mathcal{L}$ metric are not c-close to the identity. The pair $\{1,0\}$is always mapped to $\{0,1\}$, which creates a distance between an element x in $\bigoplus_\infty \mathbb{Z}_2$ and $g \circ f(x)$ that can be arbitarily large, depending on the position of the original $\{1,0\}$ pair. Any similar mapping that attempts to fix this problem still ultimately ends up mapping two distinct pairs in $\bigoplus_\infty \mathbb{Z}_2$ to the same pair in $\bigoplus_\infty \mathbb{Z}_3$, causing similar problems with c-closeness.

While this mapping ultimately does not help in proving or disproving coarse equivalence between $\bigoplus_\infty \mathbb{Z}_2$ and $\bigoplus_\infty \mathbb{Z}_3$, it does give a more intuitive sense of some of the distinctions between the two groups that might indicate that they are not coarsely equivalent.

4.2 $\bigoplus_\infty \mathbb{Z}_2$ and $\mathbb{Q}_2/\mathbb{Z}$

Proposition $(\bigoplus_\infty \mathbb{Z}_2, d\mathcal{L})$is bijectively isometric, and therefore coarsely equivalent, to $\left(\mathbb{Q}_2/\mathbb{Z}, \frac{1}{2^k}\right)$.

Proof: Let f map $\bigoplus \mathbb{Z}_2$ onto $\mathbb{Q}_2/\mathbb{Z}$, and g map $\mathbb{Q}_2/\mathbb{Z}$ onto $\bigoplus_\infty \mathbb{Z}_2$.

Define f as follows: for $\forall x \in \bigoplus_\infty \mathbb{Z}_2$, take the string $x' = \{x_1, x_2, \ldots, x_k\}$, where x_k is the last nonzero digit of x, and k is the length of the string. Measure the value of the binary string x' and take m to be the decimal representation of this value (ie, $m = 5$ for $x' = \{0, 1, 0, 1\}$). Map x to $\frac{m}{2^k}$.

Define g as the inverse of f, meaning that $\forall y \in \mathbb{Q}_2/\mathbb{Z}$, where y can be represented as $\frac{m}{2^k}$, map y to a string $x'$of length k with binary value m. Set x equal to $x'$with all positions to the right of x' filled with zeroes.

Note that the map takes every binary string to a unique element of $\mathbb{Q}_2/\mathbb{Z}$. All possible values of k are represented by the mapping, and since the last digit of the binary string must be a one, only odd values are sent to m (avoiding the possibility of mapping to an unreduced $\frac{m}{2^k}$). Also, a binary string of length k can represent any integer between 0 and $2^k - 1$ inclusive. This means that no binary strings are mapped to fractions greater than or equal to 1. Because of these properties, $g \circ f(x) = id_Y$ and $f \circ g(y) = id_X$, so the maps are 0-close to the identity.

It remains to be shown that the functions are ρ-bornologous. One can easily see that the filtrations on the two groups are actually the same definition of distance: the group generated by $\frac{1}{2^k}$ is the same as a binary string of length k. So $d\left(f\left(x\right), f\left(x'\right)\right) = d\left(x, x'\right)$, and similarly for g.

5 Polynomial Growth

5.1 Linear Growth

Lemma 5.3.1 $\mathbb{Z}$ has linear growth

Proof It's sufficient to look at the Euclidian metric, the other metrics just depend upon it. $x \in B(0, r) \Rightarrow d(0, x) = |x - 0| = |x| \leq r$. So we have $\underbrace{-r, -r+1, ..., 0, ...r-1, r}_{2r+1 elements}$.

Remark that if r is not an integer then will be looking at the ball of radius $[r]$ and in this case we'll have at least $(2r - 1)$ elements: still linear.

For any proper left invariant metric d' on $\mathbb{Z}$, $d'(0, x) \leq d'(0, 1) + ...d'(x - 1, x) = |x|d'(0, 1) = k|x|$.

5.2 Quadratic Growth

Definition 5.4.1 Let X be a group.

X is said to have quadratic growth if for any left invariant metric d on X, $|B(0,r)| \geq q(r)$ where q is a quadratic function.

Lemma 5.4.2 $\mathbb{Z} \oplus \mathbb{Z}$ has quadratic growth.

Proof Let d' be a left invariant metric on $\mathbb{Z} \oplus \mathbb{Z}$. Then

$d'((0,0),(x,y)) \leq d'((0,0),(0,1)) + ... + d'((0,y-1),(0,y))$
$+d'((0,y),(1,y)) + ...d'((x-1,y),(x,y)) = |x|d'((0,0),(0,1)) + |y|d'((0,1),(1,0))$
$\leq (|x|+|y|)\max\{u,v\}$
where $u = d'((0,0),(0,1))$ and $v = d'((0,0),(1,0))$

So calling $d(x,y) = |x|+|y|$(taxicab metric), we have $|B_d(0, r/\max\{u,v\}| \leq |B_{d'}(0,r)|$ and since $|B_d(0, r/\max\{u,v\}|$ has quadratic growth, that completes the proof.

Theorem 5.4.3 $\underbrace{\mathbb{Z} \oplus \mathbb{Z} .... \oplus \mathbb{Z}}_{n times}$, has order n^{th} order growth.

Proof Same as the the previous 2 above.

6 Property $\star$

Definition 6.1 We say that a metric space (X,d) has property $\star$ if $\forall s > 0, \exists\, x_0 \in X, r > 0$ such that $X \setminus B(x_0,r)$ is not s-connected.

Example *$\mathbb{Z}$ with any proper invariant metric has property $\star$.* That is, $\forall s > 0,\ \exists\, B(x_0,r), r > 0,\ x_0 \in \mathbb{Z}$ such that $\mathbb{Z} \setminus B(x_0,r)$ is not s-connected.

Example *$\mathbb{Z}^n$, for $n \geq 2$, however, has $\sim \star$*: $\exists s > 0$ such that $\mathbb{Z}^n \setminus B(y_0,R)$ is s-connected $\forall y_0 \in \mathbb{Z}^n, \forall R > 0$. Specifically, $\mathbb{Z}^n \setminus B$,where B is any ball, is s-connected $\forall s \geq 1$.

6.1 Asymptotic Disconnectedness

Definition 6.1.1 A metric space (X,d) is said to be asymptotically disconnected if $\forall c > 0$, X is not c-connected.

Theorem 6.1.2 For all metric spaces (X,d_X) and (Y,d_Y), $X \simeq Y \Rightarrow X$ is asymptotically disconnected $\iff Y$ is asymptotically disconnected.

Proof Assume $X \simeq Y$. Then $\exists\ f : X \to Y, g : Y \to X$ where f,g are ρ-bornologous and $f \circ g \approx_k id_y$, $g \circ f \approx_k id_x$. Assume Y is not asymptotically disconnected. Then $\exists\ c > 0$ such that $\forall y, y' \in Y,\ \exists\ \{y = y_1, y_2, \ldots y_n = y'\} \subset Y$ where $\forall\ 1 \leq i < n$, $d(y_i, y_{i+1}) \leq c$. Since $g : Y \to X$ is ρ-bornologous $\forall x, x' \in g(Y),\ \exists\ \{x = g(y_1), g(y_2), \ldots g(y_n) = x'\}$ where

$\forall\ 1 \leq i < n, d(g(y_i), g(y_{i+1})) \leq \rho(c)$. Since $g \circ f \approx_k id_x$, $\forall x \in X, x \notin g(Y)$, $\exists\ g \circ f(x) \in g(Y)$ where $d(x, g \circ f(x)) \leq k$. Let $c' = max\{\rho(c), k\}$ and X is c'-connected. A symmetric argument shows Y is asymptotically disconnected $\Rightarrow X$ is asymptotically disconnected.

6.2 $\mathbb{Z}$ and $\mathbb{Z}^n$

Theorem 2.1 $\mathbb{Z} \not\simeq \mathbb{Z}^n$.

Proof Suppose that $\mathbb{Z} \simeq \mathbb{Z}^n$. Then, $\exists\ f : \mathbb{Z} \to \mathbb{Z}^n, g : \mathbb{Z}^n \to \mathbb{Z}$ where f, g are ρ-bornologous and $f \circ g \approx_c id_{\mathbb{Z}^n}$ and $g \circ f \approx_c id_{\mathbb{Z}}$. Without loss of generality, we may assume ρ is strictly increasing, so that $t_1 > t_2 \iff \rho(t_1) > \rho(t_2)$ for $t_1, t_2 \in \mathbb{R}_+$.

Let $B = B(0, r)$ be a ball in $\mathbb{Z}$ so that $\mathbb{Z} \setminus B$ is not $(\rho(1) + c)$-connected.

Claim 1: We can find a ball, $B' \subset \mathbb{Z}^n$, with the property that for all 1-chains $\{y_1, y_2, \ldots y_k\} \subset \mathbb{Z}^n \setminus B'$, $d(g(y_i), 0) > r, \forall\, i$.

Proof of Claim: The triangle inequality provides the following result for all metric spaces $(X, d) : d(x, z) \leq d(x, y) + d(y, z) \Rightarrow d(x, y) \geq d(x, z) - d(z, y)$ and $d(z, y) \leq d(x, y) + d(x, z) \Rightarrow d(x, y) \geq d(z, y) - d(x, y)$. Putting these results together, we obtain: $d(x, y) \geq |d(x, z) - d(z, y)|$.

Hence, $\forall\, y \in \mathbb{Z}^n$: $|d(y, f(0)) - d(f \circ g(y), y)| \leq d(f \circ g(y), f(0)) \leq \rho(d(g(y), 0))$. Based on the c-closeness property of the coarse equivalence between $\mathbb{Z}$ and $\mathbb{Z}^n$, $d(f \circ g(y), y) \leq c$. Assuming $d(y, f(0)) \geq c$, we obtain $d(y, f(0)) - c \leq \rho(t)$, where $t = d(g(y), 0)$. For $g(y)$ to lie outside of $B \subset \mathbb{Z}$, it is necessary that $t > r$. This is guaranteed if we suppose $d(y, f(0)) - c > \rho(r)$. This implies that $\rho(t) > \rho(r)$, and ρ is strictly increasing implies $t > r$. Let $S = \rho(r) + c$. Thus, $d(y, f(0)) > S \iff y \in \mathbb{Z}^n \setminus B'$, where $B' = B(f(0), S) \subset \mathbb{Z}^n$.

Claim 2: $\exists\, S$ such that $S - c \leq \rho(t) \Rightarrow t > r$.

Proof of Claim: If there is no t such that $t > r$, we can choose an S so that $S - c < \rho(t)$. Then, $S - c \leq \rho(t) \Rightarrow t > r$ is vacuously true. On the other hand, if there is some t_0 such that $t_0 > r$, let $S = \rho(t_0) + c$. Then, $S - c = \rho(t_0) + c - c = \rho(t_0) \leq \rho(t)$. Since ρ is strictly increasing, this implies that $t_0 \leq t$. And since $t_0 > r$, $t > r$.

So we have shown that there does exist some ball $B' = B(f(0), S) \subset \mathbb{Z}^n$ with the property that $\forall$ 1-chains $\{y_1, y_2, \ldots y_k\} \subset \mathbb{Z}^n \setminus B'$, $d(g(y_i), 0) > r\ \forall i$. Now choose $x, x' \in \mathbb{Z} \setminus B$, which are sufficiently far from 0 to allow $f(x), f(x') \in \mathbb{Z}^n \setminus B'$. We know this is possible because the set $\mathbb{Z}$ is unbounded, while $f^{-1}(B')$ is bounded. In $\mathbb{Z}^n \setminus B'$, we can connect $f(x)$ and $f(x')$ by a 1-chain: $\{f(x) = y_1, y_2, \ldots y_k = f(x')\} \subset \mathbb{Z}^n \setminus B'$ where $\forall 1 \leq i < k$, $d(y_i, y_{i+1}) \leq 1$. Therefore, $\{g \circ f(x) = g(y_1), g(y_2), \ldots g(y_k) = g \circ f(x')\}$ is a $\rho(1)$-chain that lies in $\mathbb{Z} \setminus B$. Because $g \circ f(x) \approx_c id_{\mathbb{Z}}$, the chain $\{x, g \circ f(x), g(y_2), \ldots g(y_{k-1}), g \circ f(x'), x'\} \subset$

$\mathbb{Z} \setminus B$ is a $(\rho(1) + c)$-chain, connecting x to x'. This is a contradiction, because $\mathbb{Z}$ has property $\star$.

Thus, $\forall n \geq 2, \mathbb{Z} \not\simeq \mathbb{Z}^n$.

Property $\star$, or equivalently, $\sim \star$, is invariant under coarse equivalence for two metric spaces, in general. That is, $(X, d_x) \simeq (Y, d_y) \Rightarrow (X, d_x)$ has $\star \iff (Y, d_y)$ has $\star$.