Finitely Presented Groups in Geometry and Topology

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Abstract: In this paper, my main focus has been to prove that infinite $\bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i$ is coarsely equivalent to infinite $\bigoplus_{i=1}^{\infty}(\mathbb{Z}_3)_i$. Even though I haven't been able to find an explicit proof, I've been able to deduct some nice properties of coarse equivalence.

1 Introduction

In topology, we usually study the properties of geometric forms that remain invariant under certain transformations, as bending or stretching. So we say that two topological spaces X and Y are homeomorphic if there exists a function between those spaces that is continuous, one-to-one, and onto, and the inverse of which is continuous. Because of continuity, it's clear that one restrict itself to small scales. But what can we say about two countable groups?

2 Basic Definitons

Definition 2.1 *A* metric space *consist of a pair* (X, d)*, where* X *is a set and* $d : X \times X \to R$ *is a function, called the* metric *or* distance *such that:* $\forall x, y, z \in X$

1. $d(x, y) \geq 0$*, and* $d(x, y) = 0 \Leftrightarrow x = y : (Positive Definiteness)$
2. $d(x, y) = d(y, x) : (Symmetry)$
3. $d(x, z) \leq d(x, y) + d(y, z) : (Tringular Inequality)$

Definition 2.2 *A metric* d *on a vector space* X *is said to be* left invariant *if* $\forall x, y, z \in X$
$d(z + x, z + y) = d(x, y)$ *. On a group* $(G, *)$ *we have* $d(g, h) = d(e, g^{-1} * h)$ $\forall h, g \in G$

Definition 2.3 *A metric* d *on a metric space* X *is said to be proper if* $|B(x, r)| < \infty\ \forall x, \in X, \forall r \geq 0$
On a group $(G, *)$ *we have* $|B(g, r)| < \infty\ \forall g \in G$

Lemma 2.4 *Any proper left invariant metric on* $\mathbb{Z}$ *comes from applying a function to the usual Euclidian metric of* $\mathbb{Z}$

Proof Let d be an arbitrary proper left invariant on Z, then $d(x, y) = d(0, y-x)$ and $\exists f : \mathbb{Z} \to \mathbb{R}$ such that $d(0, s) = f(s)$. But $f(-s) = d(0, -s) = d(s, 0) = d(0, s) = f(s)$ So $d(0, y-x) = f(|y-x|)$

Lemma 2.5 *If* d *is a proper left invariant metric, Then* f *is increasing,*$f(0) = 0$ *and* f *is concave* $\Rightarrow$ $f(d)$ *is a proper left invariant metric .*

Proof For, we just need to check the triangle inequality: $f(d(x, z)) \leq f(d(x, y) + d(y, z)) \leq f(d(x, y)) + f(d(y, z))$

Lemma 2.6 *For any set* A *and a metric space* (X, d)*, we can define a metric on* A *by defining a injective function* $f : A \to X$ *by* $d'(x, y) = d(f(x), f(y))$ *where* $x, y \in A$. d' *is proper left invariant* $\Leftrightarrow$ f *is linear*

Proof Since f is injective then $x \neq y$ and $d'(x, y) = 0$ is remote. we just need to check the triangle inequality again: $d'(x, z) = d(f(x), f(z)) \leq d(f(x), f(y)) + d(f(y), f(z)) = d'(x, y) + d'(y, z)$
f linear $\Leftrightarrow f(x-y) = f(x) - f(y) \Leftrightarrow d'(x, y) = d(f(x), f(y)) = d(0, f(y) - f(x)) = d(0, f(y-x)) = d'(0, y-x)$

Definition 2.7 *A metric space* (X, d) *is called* ultrametric *if*
$\forall x, y, z \in X \;\; d(x, z) \leq \max\{d(x, y), d(y, z)\}$

Properties of ultrametrics $\forall x, y, x, \in X$,$r, s \in R$

- Every triangle is isosceles;i.e. $d(x, y) = d(y, z)$ or $d(x, z) = d(y, z)$ or $d(x, y) = d(z, x)$.
- Every point inside a ball is its center; i.e. if $d(x, y) < r$ then $B(x; r) = B(y, r)$.
- Intersecting balls are contained in each other; i.e. if $B(x; r) \cap B(y; s)$ is non-empty then either $B(x, r) \subseteq B(y; s)$ or $B(y, s) \subseteq B(x, r)$.

Definition 2.8 *A map* $r : X \to X$ *is called a* retraction *if* $r(x) = x$, $\forall x \in r(X)$. *A subspace* $A \subseteq X$ *is called a* retraction of X *if there exists a retraction on* X *onto* A.

Example $\mathbb{Z} \hookrightarrow \mathbb{R}, z \mapsto z$ is a retraction

Definition 2.9 *A map* $f : X \to Y$ *of metric spaces is called lipschitz if there is a constant* $\lambda > 0$ *such that the inequality* $d_Y(f(x), f(y)) \leq \lambda . d_X(x, y)$ *holds*$\forall x, y \in X$. f *is called* $\lambda - lipschitz$ *if we need to specify the constant* λ. f *is called* $\lambda - bi - lipschitz$ *if both* f *and* f^{-1} *are* $\lambda - lipschitz$

3 Growth of a group

Definition 3.1 *Let X be a group $f : \mathbb{R}(or\mathbb{Z}) \to \mathbb{Z}, r \mapsto |B_d(0,r)|$ is called the growth function of G.d is assumed to be a proper left invariant .*

Lemma 3.2 *$\mathbb{Z}$ has linear growth*

Proof It's sufficient to look at the Euclidian metric, the other metrics just depend upon it. $x \in B(0,r) \Rightarrow d(0,x) = |x - 0| = |x| \leq r$. So we have $\underbrace{-r, -r+1, ..., 0, ...r-1, r}_{2r+1 elements}$. Remark that if r is not an integer then will be looking at the ball of radius $[r]$ and in this case we'll have at least $(2r-1)$ elements: still linear.
For any proper left invariant metric d' on $\mathbb{Z}$, $d'(0,x) \leq d'(0,1) + ...d'(x-1,x) = |x|d'(0,1) = k|x|$

Definition 3.3 *Let X be a group. X is said to have quadratic growth if for any left invariant metric d on X, $|B(0,r)| \geq q(r)$ where q is a quadratic function.*

Lemma 3.4 *$\mathbb{Z} \oplus \mathbb{Z}$ has quadratic growth.*

Proof Let d' be a left invariant metric on $\mathbb{Z} \oplus \mathbb{Z}$ then
$d'((0,0),(x,y)) \leq d'((0,0),(0,1)) + ... + d'((0,y-1),(0,y))$
$+d'((0,y),(1,y)) + ...d'((x-1,y),(x,y)) = |x|d'((0,0),(0,1)) + |y|d'((0,1),(1,0))$
$\leq (|x| + |y|) \max\{u,v\}$
where $u = d'((0,0),(0,1))$ and $v = d'((0,0),(1,0))$
So calling $d(x,y) = |x| + |y|$(taxicab metric), we have $|B_d(0, r/\max\{u,v\}| \leq |B_{d'}(0,r)|$ and since $|B_d(0, r/\max\{u,v\}|$ has quadratic growth, that completes the proof.

Theorem 3.5 *$\underbrace{\mathbb{Z} \oplus \mathbb{Z} .... \oplus \mathbb{Z}}_{ntimes}$, has order n^{th} order growth.*

Proof Same as the the previous 2 above

Conjecture: $\bigoplus_{i=1}^{\infty}(\mathbb{Z})_i$ has exponential growth. Incomplete proof:
$\bigoplus_{i=1}^{\infty}(\mathbb{Z})_i = \bigoplus_{i=1}^{n}(\mathbb{Z})_i \oplus \bigoplus_{i=1}^{n}(\mathbb{Z})_i \oplus \bigoplus_{i=1}^{n}(\mathbb{Z})_i \oplus ....$ For any n, which would mean that it grows faster than any polynomial.

4 Coarse equivalence

Definition 4.1 *We call a map $f : X \to Y$ of metric spaces bornologous (or large scale uniform) if there is a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d_Y(f(x), f(y)) \leq \rho(d_X(x,y))$, $\forall x, y \in X$. Then f is said to be $\rho - bornologous$*

Remark $\rho - bornologous$ functions are just generalization of lipschitz functions since for the lipschitz case $\rho(t) = \lambda.t$

Example The application $f : \mathbb{Z} \to \mathbb{Z}, x \mapsto ax + b$ is $\rho - bornologous$ where $\rho(t) = kt, \forall k \geq |a|$ since $d(f(x), f(y)) = d(ax + b, ay + b) = |a|d(x, y) \leq kd(x, y)$

Remark that function of the type $f(x) = x^{\alpha}$ where $\alpha \leq 1$ are $\rho - bornologous$

Remark $\rho - bornogolous$ is a very weak property since we can use fast growing functions such as e^x

Definition 4.2 *Two maps*$f : X \to Y$ *and* $g : X \to Y$ *are said to be* $C - close$ *if there exist* $C > 0$ *such that* $d_Y(f(x), g(x)) \leq C$, $\forall x \in X$

Definition 4.3 *Two Metric spaces* X *and* Y *are said to be coarsely equivalent if there exist* $f : X \to Y$ *and* $g : Y \to X$ *that are* $\rho - bornologous$ *and there exist* $C > 0$ *such that* $d_X(x, gof(x)) \leq C$ *and* $d_Y(y, fog(y)) \leq C$ *are said to be* $C - close$ *if there exist* $C > 0$ *such that* $d(f(x), g(x)) \leq C$, $\forall x \in X$

Example $\mathbb{Z}$ is coarse equivalent to $\mathbb{R}$
For, let $f : \mathbb{Z} \hookrightarrow \mathbb{R}, n \mapsto n$ and $g : \mathbb{R} \to \mathbb{Z}, x \mapsto int(x)$
these two functions are clearly bornologous and $d_X(n, gof(n)) = d_X(n, n) = 0 \leq 1$
$d_Y(x, int(x)) \leq 1$

An intuitive understanding of Coarse equivalence is to see if two spaces that look different from a small scale, look the same if we very far away.

Proposition 4.4 *Coarse equivalence is an equivalence relation.*

Proof *Reflexivivity.* If (X, d) is a metric space Then define $f : X \to X, x \mapsto x$ and $g = f$ for all $x \in X$.
$\rho - bornologous$
$d(f(x), f(x')) = d(x, x')$ and same for $d(g(x), g(x')) = d(x, x')$. Setting $\rho(t) = t$, f is $\rho - bornologous$
It is clear that $f \circ f = f \circ g = g \circ f = id_X$ are C-close to the identity $\forall C \geq 0$.

Symmetry. $X \simeq^{coarse} Y \Rightarrow \exists f : X \to Y$ and $g : Y \to X$ such that f, g are ρ-bornologous and $\exists C > 0$ where $f \circ g, g \circ f$ are c-close to the identity map. Just switching the role of f and g gives $Y \simeq^{coarse} X$.
Transitivity. Suppose $X \simeq^{coarse} Y$ and $Y \simeq^{coarse} Z$ Then $\exists f_1 : X \to Y, g_1 : Y \to X, f_2 : Y \to Z, g_2 : Z \to Y$ where f_1, g_1, f_2, g_2 are ρ-bornologous for some $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and $\exists C > 0$ such that $d(x, g_1 \circ f_1(x)) < C$, $d(y, f_1 \circ g_1(y)) < C$ and $d(y, g_2 \circ f_2(y)) < C$ and $d(z, f_2 \circ g_2(z)) < C$.

We need to find two functions $f : X \to Z$ and $g : Z \to X$ that are $\rho - bornologous$ and such that $d_X(x, gof(x)) \leq C'$ and $d_Z(z, fog(z)) \leq C'$ for some C'. Let $f = f_2 \circ f_1$ and $g = g_1 \circ g_2$

$\rho - bornologous$
$d(f(x), f(x')) = d(f_2 \circ f_1(x), f_2 \circ f_1(x')) = d(f_2(f_1(x)), f_2(f_1(x'))) \leq \rho(d(f_1(x), f_1(x'))) \leq \rho(\rho(d(x, x'))) = \rho \circ \rho(d(x, x'))$. since ρ is increasing.

Similarly, $d(g(z), g(z')) = d(g_1 \circ g_2(z), g_1 \circ g_2(z')) = d(g_1(g_2(z)), g_1(g_2(z'))) \leq \rho(d(g_2(z), g_2(z'))) \leq \rho(\rho(d(x, x'))) = \rho \circ \rho(d(x, x'))$. since ρ is increasing.

$C - closeness$

$d(x, g \circ f(x)) = d(x, g(f(x))) = d(x, g_2 \circ g_1 \circ f_2 \circ f_1(x)) \leq d(x, g_1 \circ f_1(x)) + d(g_1 \circ f_1(x), g_2 \circ g_1 \circ f_2 \circ f_1(x))$(By the triangle inequality,)
And since $Y \simeq Z$ then $d(y, g_2 \circ f_2(y)) \leq c$, in particular for $y = f_1(x)$. So $d(f_1(x), g_2 \circ f_2 \circ f_1(x)) \leq c\,, \forall x \in X$. Since g_1 is $\rho - bornologous$, applying it to the left side of the previous inequality we have $d(g_1(g_1(f_1(x)), g_1(g_2 \circ f_2 \circ f_1(x))) \leq \rho(d(f_1(x), g_2 \circ f_2 \circ f_1(x)) \leq \rho(c)$.
So $d(x, g_1 \circ g_2 \circ f_2 \circ f_1(x)) \leq d(x, g_1 \circ f_1(x)) + d(g_1 \circ f_1(x), g_2 \circ g_1 \circ f_2 \circ f_1(x)) \leq \rho(c) + c$.
Let $C' = \rho(C) + C$ then $d(x, g \circ f(x)) \leq C'$
Similarly we get that $d(z, g \circ f(z)) \leq C'$

Proposition 4.5 *Let X and Y be two metric spaces. Then $X \simeq^{coarse} Y \Rightarrow$ the inverse image of each bounded under f set in Y is bounded (Properness)*

Proof Suppose $X \cong^{coarse} Y$ then there exist $f : X \to Y$ and $g : Y \to X$ such that f and g are $\rho - bornologous$ and $d_X(x, gof(x)) \leq C$ and $d_Y(y, fog(y)) \leq C$
Let $S = \bigcup\{y_i\}$ be an arbitrary closed set of Y $f^{-1}(\{y_i\}) = \{x \in X | f(x) = y_i\}$.
If $f^{-1}(\{y_i\})$ contains only one element, then nothing to prove.
So suppose $f^{-1}(\{y_i\})$ contains at least two elements x and x'.
If $f^{-1}(S)$ is not bounded, then $\forall k > 0, d(x, x') > k$ but $d(x, x') \leq d(x, g \circ f(x)) + d(x', g \circ f(x'))$ since $g \circ f(x) = g(f(x)) = g(f(x')) = g \circ f(x')$
So $d(x, x') \leq d(x, g \circ f(x)) + d(x', g \circ f(x')) \leq C + C = 2C$ Contradiction since $d(x, x')$ supposed to be greater than any $k > 0$.

Definition 4.6 *Two Metric spaces X and Y are said to be bijectively coarse equivalent if there exist $f : X \to Y$ and $g : Y \to X$ that are bijective,$\rho - bornologous$ and there exist $C > 0$ such that $d_X(x, gof(x)) \leq C$ and $d_Y(y, fog(y)) \leq C$ are said to be $C-close$ if there exist $C > 0$ such that $d(f(x), g(x)) \leq C, \forall x \in X$*

5 The d_L Metric

Let $\bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i$ so elements in X are under the form $(a_1 a_2, ..., a_n, 0, 0, 0, ...)$ where $a_i = 0, 1$ which means that there is some n after which the $a_i's$ are zero. We can then compare two elements in X by looking at the largest index at which they different, this index will be said to the distance between these two points in X.

Example let $a = (011101000...000...)$ and $b = (10010001000....000...)$
then $d(a, b) = 8$

Definition 5.1 *We define $d_L(a, b) = \max\{i | a_i \neq b_i\}$.*

Let G be a loacally finite group, there is way of defining a proper left invariant metric on G by considering a filtration L of G. So if consider a filtration L of G

by subgroups $L = \{1 \subset G_1 \subset G_2 \subset G_3 \subset ...\}$, we define the metric d_L associated with this filtration as:$d_L(x, y) = \min\{i|x^{-1}y \in G_i\}$.

Proposition 5.2 *d_L is an ultrametric*

Proof Let's use the fact that any triangle in an ultrametric space is isosceles Let consider $a = (a_1, a_2, ..., a_n, 0, ...)$, $b = (b_1, b_2, ..., b_m, 0, ...)$,$c = (c_1, c_2, ..., c_l, 0, ...)$
We need to show that if $d_L(a, b) \geq d_L(b, c)$ Then $d_L(a, b) = d_L(a, c)$ or $d_L(a, b) = d_L(b, c)$
So $_L(a, b) = \max\{n, m\}$ and $d_L(b, c) = \max\{m, l\}$ and then $\max\{n, m\} \geq \max\{m, l\}$
We decompose this inequality in two cases:
Case 1: If $\max\{n, m\} = n$ Then $n \geq l \Rightarrow d_L(a, c) = n = d_L(a, b)$
Case 2: If $\max\{n, m\} = m$ Then $\max\{n, m\} \geq \max\{m, l\} \Rightarrow m \geq l$ otherwise we will have $\max\{m, l\} = l$ and contradiction from $\max\{n, m\} = m \geq \max\{m, l\} = l$.
So $\max\{m, l\} = m = \max\{n, l\}$ and then $d_L(a, b) =_L (b, c)$.

Example A filtration L of $\bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i$ is $L = \mathbb{Z}_2 \subset \mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset \mathbb{Z}_2 \oplus Z_2 \oplus \mathbb{Z}_2 \subset ....$

let's find the Growth function of $\bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i$: $B(0, r) = \{a \in \bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i | d_L(0, a) \leq r\}$
$d_L(0, (a_1, a_2, ..., a_n, 0, 0, 0, ...)) \leq r \Rightarrow n \leq r \Rightarrow |B(0, r)| = 2^{[r]}$since each time there is 2 ways of picking up the $a_i's$.

Let's find the Growth function of $\bigoplus_{i=1}^{\infty}(\mathbb{Z}_3)_i$: $B(0, r) = \{a \in \bigoplus_{i=1}^{\infty}(\mathbb{Z}_3)_i | d_L(0, a) \leq r\}$
$d_L(0, (a_1, a_2, ..., a_n, 0, 0, 0, ...)) \leq r \Rightarrow n \leq r \Rightarrow |B(0, r)| = 3^{[r]}$ since each time there is 3 ways of picking up the $a_i's$.

Proposition 5.3 *If (X, d) is a group with d the taxicab metric and d' is any proper left invariant metric other than d then $X, d) \simeq^{coarse} (X, d')$*

Proof Define $f : (X, d) \to (X, d'), x \mapsto x$ and $g : (X, d') \to (X, d), x \mapsto x$
$\rho - bornologous$
Let $\lambda_1 = max\{d(0, e_i) : e_i, generators \in X\}$ and $\lambda_2 = \frac{1}{\lambda_1}$
Then from the proof of **lemma 3.4**
$d'(f(x), f(y)) = d(x, y) = d(0, y - x) \leq \lambda_1.d(0, y - x) = \lambda_1.d(x, y)$
Similarly,
$d(g(x), g(y)) = d(x, y) = d(0, y - x) \leq \lambda_2.d'(0, y - x) = \lambda_2.d'(x, y)$
So f is $\rho_1 - bornologous$ and g is $\rho_2 - bornologous$ where $\rho_1(t) = \lambda_1.t$ and $\rho_2(t) = \lambda_2.t$
$C - closeness$
$d(x, g \circ f(x)) = d(x, g(x)) = d(x, x) = 0$ and $d'(x, f \circ g(x)) = d'(x, f(x)) = d'(x, x) = 0$
So it's sufficient to take $C = 0$

Corollary 5.4 *If G is a group and $L = G_0 \subset G_1 \subset ...$ and L' $= G'_0 \subset G'_1 \subset ...$ two filtrations of G, then (G, d_L) and $(G, d_{L'})$ are bijectively coarse equivalent.*

Proof $(G, d_L) \simeq^{coarse} (G, d)$ and $(G, d) \simeq^{coarse} (X, d_{L'})$
Now since $\simeq^{coarse}$ is an equivalence relation(**proposition 4.4**), we just proved that $(G, d_L) \simeq^{coarse} (G, d_{L'})$ It's bijective since we're just using the identity map on G

6 Problems and Properties

6.1 $\bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i$ and $\bigoplus_{i=2}^{\infty}(\mathbb{Z}_2)_i$ coarsely?

Proof Define
$f : \bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i \to \bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i$ $a = (a_,1a_2, ..., a_n, 0, 0, 0, ...) \mapsto (0, a_2, ..., a_n, 0, 0, 0, ...)$
$g : \bigoplus_{i=2}^{\infty}(\mathbb{Z}_2)_i \hookrightarrow \bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i$ the canonical inclusion.
$\rho - bornologous$
$d(f(a), f(b)) = d((0, a_2, ..., a_n, 0, 0, 0, ...), (0, b_2, ..., b_m, 0, 0, 0, ...)) \leq \max\{m, n\}$
And $d(a, b) = d((a_1a_2, ..., a_n, 0, 0, 0, ...), (b_1, b_2, ..., b_m, 0, 0, 0, ...)) = \max\{n, m\}$
So taking $\rho(t) = t \Rightarrow d(f(a), f(b)) \leq \max\{m, n\} \leq \rho(\max\{m, n\}) = \rho(d(a, b))$
$d(g(a), g(b)) = d((0, a_2, ..., a_n, 0, 0, 0, ...), (0, b_2, ..., b_m, 0, 0, 0, ...)) = d(a, b)$
So by taking $\rho(t) = t$ we have $d(g(a), g(b)) \leq \rho(d(a, b))$. $C - closeness$
$d(a, g \circ f(a)) = d((a_1, a_2, ..., a_n, 0, 0, 0, ...), (0, a_2, ..., a_n, 0, 0, 0, ...)) = 1$
$d(a', f \circ g(a')) = d((0, a'_2, ..., a'_n, 0, 0, 0, ...), (0, a_2, ..., a_n, 0, 0, 0, ...)) = 0$
So by taking $C = 1$ we have $g \circ f \approx_C Id_{\bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i}$ and $f \circ g \approx_C Id_{\bigoplus_{i=2}^{\infty}(\mathbb{Z}_2)_i}$

6.2 If H_1, H_2 are two finite groups and G a group Then $H_1 \oplus G \simeq^{coarse} H_2 \oplus G$

Proof Let $f : H_1 \oplus G \to H_2 \oplus G, (h_1, g) \mapsto (0, g)$ and
$g : H_2 \oplus G \to H_1 \oplus G, (h_2, g) \mapsto (0, g)$ where $h_1 \in H_1, h_2 \in H_2, g \in G$
$\rho - bornologous$
$d(f(h_1, g), (h'_1, g)) = d((0, g), (0, g)) = 0 \leq d((h_1, g), (h'_1, g)), \forall (h_1, g), (h'_1, g) \in H_1 \oplus G$ and similarly
$d(f(h_2, g), (h'_2, g)) = d((0, g), (0, g)) = 0 \leq d((h_2, g), (h'_2, g)), \forall (h_2, g), (h'_2, g) \in H_2 \oplus G$
So taking $\rho(t) = t$ is more than sufficient.
$C - closeness$
Let's denote $D_1 = max\{d(h_1, h'_1) : h_1, h'_1 \in H_1\}$ and $D_2 = max\{d(h_2, h'_2) : h_2, h'_2 \in H_2\}$ which we know exist since and H_1, H_2 are two finite groups.
Let $C = max\{D_1, D_2\}$ then
$d(h_1, g \circ f(h_1)) \leq C$ since $h_1, g \circ f(h_1) \in H_1$.
Similarly $d(h_2, f \circ g(h_2)) \leq C$ since $h_2, f \circ g(h_2) \in H_2$

So, $H_1 \oplus G \simeq^{coarse} H_2 \oplus G$

Example $\mathbb{Z}_a \oplus \bigoplus_{i=1}^{\infty}(\mathbb{Z}_c)_i \simeq^{coarse} \mathbb{Z}_b \oplus \bigoplus_{i=1}^{\infty}(\mathbb{Z}_c)_i$

Proof $\mathbb{Z}_a = \{\bar{0}, \bar{1}, ...\overline{a-1}\}$ and $\mathbb{Z}_a = \{\bar{0}, \bar{1}, ...\overline{b-1}\}$ The rest is the same as the previous proof by letting $\bigoplus_{i=1}^{\infty}(\mathbb{Z}_c)_i = G$.

Example $\bigoplus_{i=1}^{n}(\mathbb{Z}_a)_i \oplus G \simeq^{coarse} \bigoplus_{i=1}^{m}(\mathbb{Z}_b)_i \oplus G$

Proof Same proof by letting $H_1 = \bigoplus_{i=1}^{n}(\mathbb{Z}_a)_i$ and $H_2 = \bigoplus_{i=1}^{m}(\mathbb{Z}_b)_i$

Example $\mathbb{Z}_3 \oplus \bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i \simeq^{coarse} \bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i$

Proof Same by letting $H_1 = \bigoplus_{i=1}^{n}(\mathbb{Z}_3)_i$ and $H_2 = \{0\}$ and $G = bigoplus_{i=1}^{\infty}(\mathbb{Z}_2)_i$

6.3 [Dr. Conant: Property $\star$]: A space X is said to have property $\star$ if $\forall c > 0, \exists R > 0, \exists x_0 \in X$ such that $X \backslash B_R(x_0)$ is not c-connected

Proposition 6.1 (Dr. Conant) *: If X has property $\star \Rightarrow X$ is unbounded*

Proof : Suppose X is not unbounded, then $D = diam(X) < \infty$, So $X \backslash B_R(x_0) \subset X$ for any $B_R(x_0)$ and then $X \backslash B_R(x_0)$ is $D - connected.$ Remark that I am assuming that $X \backslash B_R(x_0) \neq \emptyset$.

6.4 Theorem [Dr. Conant]: $\mathbb{Z}$ is not coarsely equivalent to $\mathbb{Z} \oplus \mathbb{Z}$

Proof (Dr.Conant) Suppose otherwise. Then we have maps $f : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ and $g : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f$ and $f \circ g$ are c-close to the identity and f and g are $\rho - bornologous$. Let B be a ball of radius R centered at 0 in $\mathbb{Z}$ such that $\mathbb{Z}$ B is not $(\rho(1) + c) - connected.$

Claim 1: There exists an S such that $\rho(t) \geq S - c \Rightarrow t > R$.

There are two cases. If there are no t such that $\rho(R) < \rho(t)$, then choose S so that $S > \rho(R) + c$. Then there are no t satisfying $\rho(t) \geq S - c$, so that the implication $\rho(t) \geq S - c \Rightarrow t > R$ is vacuously true. On the other hand, if there is some t_0 such that $\rho(t_0) > \rho(R)$, then let $S = \rho(t_0) + c$. Then if $\rho(t) \geq S - c$, that implies that $\rho(t) \geq \rho(t_0) > \rho(R)$ which implies that $t > R$ as desired since ρ is an increasing function.

Let $B' \subset \mathbb{Z} \oplus \mathbb{Z}$ be a ball around $f(0)$ of radius S, as in the previous claim.

Claim 2: We have $g(\mathbb{Z} \oplus \mathbb{Z} \backslash B') \subset \mathbb{Z} \backslash B$. Let $y \in \mathbb{Z} \oplus \mathbb{Z} \backslash B'$, then $d(y; f(0)) > S$. Now $\rho(d(g(y); 0)) \geq d(fg(y); f(0)) \geq |d(fg(y); y) - d(y; f(0))| \geq S - c$

So by **claim 1**, with $t = d(g(y); 0)$, we have $d(g(y); 0) > R$.

So $g(y) \in \mathbb{Z} \backslash B$.

Now choose $x, x' \in \mathbb{Z}$ which cannot be connected by a $c + \rho(1)) - chain$, and which are sufficiently far from $\{0\}$ such that $f(x); f(x') \notin B'$. (Since B' is bounded, we know that $f^{-1}(B)$ is a bounded set.) In $g(\mathbb{Z} \oplus \mathbb{Z} \backslash B')$, we can connect $f(x); f(x')$ by a 1-chain:

$f(x) = y_0; y_1; y_2; ...; y' = f(x')$ where $d(y_i; y_{i+1}) \leq 1$. Then we know

$g(f(x)); g(y_0); ...; g(y_{l-1}); g(f(x_0))$ is a $\rho(1) - chain$, which lies in $\mathbb{Z} \backslash B$.

Prepending x to the beginning and appending x' to the end, we get a $c + \rho(1)$ chain connecting x to x', which is a contradiction.

6.5 Question[Dr. Conant]: Is the growth function conserved by Coarse equivalence? i.e. $X \simeq^{coarse} Y$ and X has $n^{th} - order$ growth degree $\Rightarrow Y$ has $n^{th} - order$ growth.

It works for $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ since $\mathbb{Z}$ has linear growth and $\mathbb{Z} \oplus \mathbb{Z}$ has quadratic growth and that they are not coarsely equivalent. But about in general?

6.6 Question [Dr. Conant]: Is Property $\star$ conserved by Coarse equivalence? i.e. $X \simeq^{coarse} Y$ and X has Property $\star \Rightarrow Y$ has Property $\star$.

It works for $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ since $\mathbb{Z}$ has Property $\star$ and $\mathbb{Z} \oplus \mathbb{Z}$ doesn't have Property $\star$ and that they are not coarsely equivalent. But about in general?

References

[1] N. Brodskiy, *LaTeX : Asymptotic Dimenstion of Groups.*

[2] N. Brodskiy, J. Dydak, J. Higes and A. Mitra *LaTeX : Dimension zero at all scales.*