

Knots 3

Exercise 1. Recall that the group of the trefoil is isomorphic to the group $\langle u, v \mid u^3 = v^2 \rangle$. Find a homomorphism of this group onto the alternating group A_4 .

(A_4 is the group of order 12 consisting of all even permutations on four symbols, specifically the identity element, the eight 3-cycles $(1\ 2\ 3)$, $(1\ 2\ 4)$, $(1\ 3\ 2)$, $(1\ 3\ 4)$, $(1\ 4\ 2)$, $(1\ 4\ 3)$, $(2\ 3\ 4)$, $(2\ 4\ 3)$, and the three “double transpositions” $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, $(1\ 4)(2\ 3)$.)

Exercise 2. Draw a diagram of the figure-eight knot, and from it obtain a presentation of the group of this knot. Eliminate some of the generators to obtain a presentation with only two generators.

Exercise 3. All knots of up to 7 crossings are so-called *2-bridged knots*, meaning that the knot curve can be arranged so that it has exactly two local maxima with respect to the z -coordinate. Show that the first four knots in Table 1 (from Wednesday) are two-bridged.

Exercise 4. Find a diagram of the trefoil knot with only two bridges (a bridge is a segment of knot diagram that goes over any number ≥ 1 of strands, but which never goes under a strand.) **Challenge question:** do the same for the figure-eight knot. *Hint:* for the trefoil you’ll need 4 crossings, and for the figure-eight you’ll need 8 crossings.

Exercise 5. Explain why a 2-bridged knot always admits a diagram with two bridges, and why (therefore) its group can always be generated by two elements. *Hint:* arrange the knot curve so that it has two local maxima, and then stand above it.

Remark. 2-bridged knots were completely classified by H. Schubert in the 1950’s. On the other hand, 3-bridged knots remain unclassified to this day.

Remark. Knot groups can be represented faithfully by 2×2 matrices over the complex numbers. For example, the group of the figure-eight knot is isomorphic to the group generated by the two matrices

$$\begin{bmatrix} 1 & -\omega \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ \omega & 1 \end{bmatrix}, \quad \text{where } \omega = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

This representation of the figure-eight knot group was discovered by R. Riley, who realized that it implied the existence of a hyperbolic geometric structure on the complement of the knot. W. Thurston, after a casual conversation with Riley over coffee, then made the profound observation that *all* prime knots, apart from certain specific exceptions, had hyperbolic structures on their complements.