

Knots 2

These notes aren't intended to be self-sufficient, but rather to supplement the lecture.

The Fundamental group of a topological space. A *path in a space* X is a continuous function from the unit interval $[0, 1]$ into X . We can think of the interval $[0, 1]$ as a unit interval of time (*e.g.* 1 hour). A path $f : [0, 1] \rightarrow X$ then describes a journey in X , as it tells us our position in X at each point in time.

If the path f begins and ends at the same point $x_0 \in X$, we say that f is a *loop* in X based at x_0 .

Definition. Let f, g be paths in X that both start at $a \in X$, and both finish at $b \in X$. We say that f, g in X are *path homotopic* if there exists a continuously parametrized family of paths f_s in X ($0 \leq s \leq 1$) such that $f_0 = f$, $f_1 = g$ and $f_s(0) = a$, $f_s(1) = b$ for each $t \in [0, 1]$.

The way to think of this is that we can deform the path f continuously, keeping its endpoints fixed, so that it ends up being g .

Definition. Let X be a topological space, and let x_0 be a fixed point in X . The *fundamental group of X based at x_0* , denoted $\pi_1(X, x_0)$, is a group described as follows:

- (i) The elements of $\pi_1(X, x_0)$ are equivalence classes of loops in X based at x_0 , under the relation of path homotopy. (This just means that we regard two loops as being “the same” if one can be continuously deformed to the other, keeping the ends fixed.)
- (ii) (Equivalence classes of) loops are “multiplied together” as follows:

$$(f * g)(t) = \begin{cases} f(2t) & (0 \leq t \leq \frac{1}{2}) \\ g(2t - 1) & (\frac{1}{2} \leq t \leq 1) \end{cases}$$

Thus the combined path $f * g$ can be interpreted as follows: do f at double speed for the first half-hour, and then do g at double speed for the remaining half-hour.

The identity element of the group $\pi_1(X, x_0)$ is represented by the “lazy” path where we stay at the basepoint for the entire time period, and the inverse of an element is represented by a path that’s the reverse of the original, *i.e.* run the film backwards.

Note. If the space X is path connected, different choices of basepoint don’t affect the fundamental group in any essential way. We just need to choose a basepoint.

Example. Take X to be the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and take the basepoint x_0 to be the real number 1. Then $\pi_1(X, x_0) = \pi_1(S^1, 1)$ is isomorphic to the familiar additive group of integers, $(\mathbb{Z}, +)$. All that matters is the net number of times we travel around the circle counterclockwise (clockwise counts negatively.)

Example. Let X be the unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$, and let x_0 be the origin. Then $\pi_1(X, x_0)$ is trivial.

Example. Consider the *unit 2-sphere* $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$, and let x_0 be any point of S^2 . Then $\pi_1(S^2, x_0)$ is trivial. (This takes a bit of proving, as there exist paths that fill up the whole of S^2 .)

Example. Consider the *torus* $T = S^1 \times S^1$, with basepoint $x_0 = (1, 1)$.
 $\pi_1(T, x_0) \cong \mathbb{Z} \times \mathbb{Z}$.

Example. Let X be constructed by glueing two circles together at a single point, so as to make an “8”, and let the basepoint x_0 be the point where they’re glued. This space is called a *wedge of two circles* and is denoted $S^1 \vee S^1$. Let a be represented by a loop travelling once around the first circle, and let b be represented by a loop travelling once around the second circle. Then $\pi_1(S^1 \vee S^1, x_0)$ is the *free group* generated by a, b , i.e.

$$\pi_1(S^1 \vee S^1, x_0) \cong \langle a, b \mid \rangle .$$

Example. Let X be the *projective plane* \mathbb{P}^2 , and let x_0 be any point in X . Then
 $\pi_1(\mathbb{P}^2, x_0) \cong \mathbb{Z}_2$,
i.e. the fundamental group has order 2.

Fact. Let G be an arbitrary group. Then G is isomorphic to the fundamental group of some space.

Definition. Let K be a knot. The *group of K* is the fundamental group of the space $\mathbb{R}^3 - K$ (it doesn’t matter which point we choose as basepoint x_0 .)

We can read off a presentation of the group of K from a diagram of K , in particular the *Wirtinger presentation*.

The group of the unknot is isomorphic to \mathbb{Z} , but the group of any non-trivial knot is infinite and non-abelian. The first proof that the group of a non-trivial knot can’t be \mathbb{Z} appeared in 1957 and is very hard (C.D. Papakyriakopoulos, Dehn’s Lemma).

The perfidious lemma of Dehn
Was every topologist’s bane
’Til Christos Papa-
kyriakopou-
los proved it without any strain.

The knot group is a fantastically powerful invariant: in 1988 it was established, as a result of work by various authors (principally C.McA. Gordon and J. Luecke), that any two distinct prime knots have groups that are not isomorphic (this is an order of magnitude harder to prove even than Dehn's Lemma.)

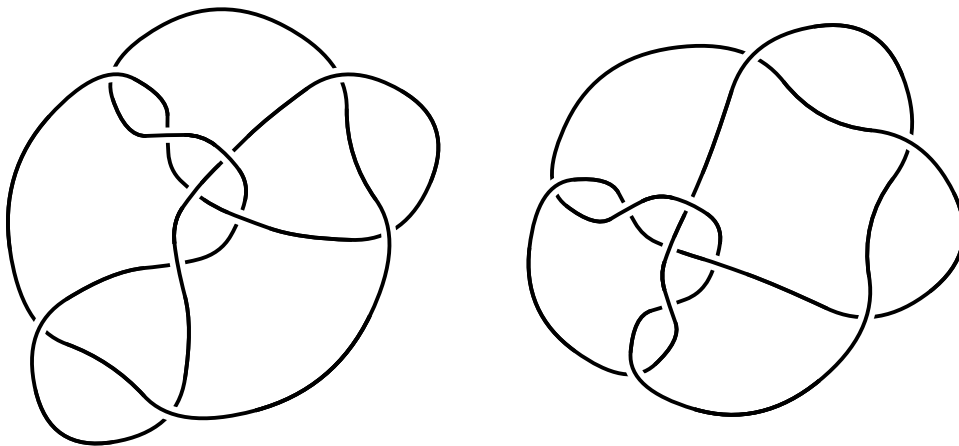
However, it isn't necessarily easy in general to decide whether two given knot groups are non-isomorphic, but in practice one can usually do it, using for example *homomorphisms of the knot group onto finite groups*.

Perhaps the simplest way in which homomorphisms of the knot group arise is with our old friend *n-coloring*.

Theorem. A knot K admits an n -coloring if and only if the group of K admits a homomorphism onto a non-abelian subgroup of the *dihedral group* D_{2n} (consisting of the $2n$ symmetries of a regular n -gon.)

For example, a knot K can be 3-colored if and only if the non-abelian group D_6 (of order 6) is a homomorphic image of the group of K .

If we have two knots K_1, K_2 where the group of K_1 admits a homomorphism onto a certain finite group, but the group of K_2 doesn't, then the groups of K_1, K_2 can't be isomorphic, and we deduce that the knots K_1, K_2 aren't equivalent.



The notorious Perko pair that eluded knot theorists for 80 years