Knots 1

A *classical knot* is a smooth or piecewise linear simple closed curve in 3-dimensional Euclidean space $\mathbb{R}^3$. Knots K_1, K_2 are *equivalent* if there exists a continuously parametrized family of homeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ $(0 \le t \le 1)$ such that h_0 is the identity map and h_1 maps K_1 to K_2.

Rather more informally, if we "thicken" K_1, K_2 so that they look like (knotted) loops of rope, then K_1 is equivalent to K_2 iff K_1 can be continuously deformed to K_2.

It can happen that a knot is equivalent to its *mirror image*, in which case the knot is said to be *amphicheiral*.

There exist *tables of knots* up to 17 crossings. In the tables we don't include mirror images, but a (small) list of the amphicheiral knots is given separately. Here are the *prime* knots up to 7 crossings; only the second and fifth of these knots are amphicheiral.

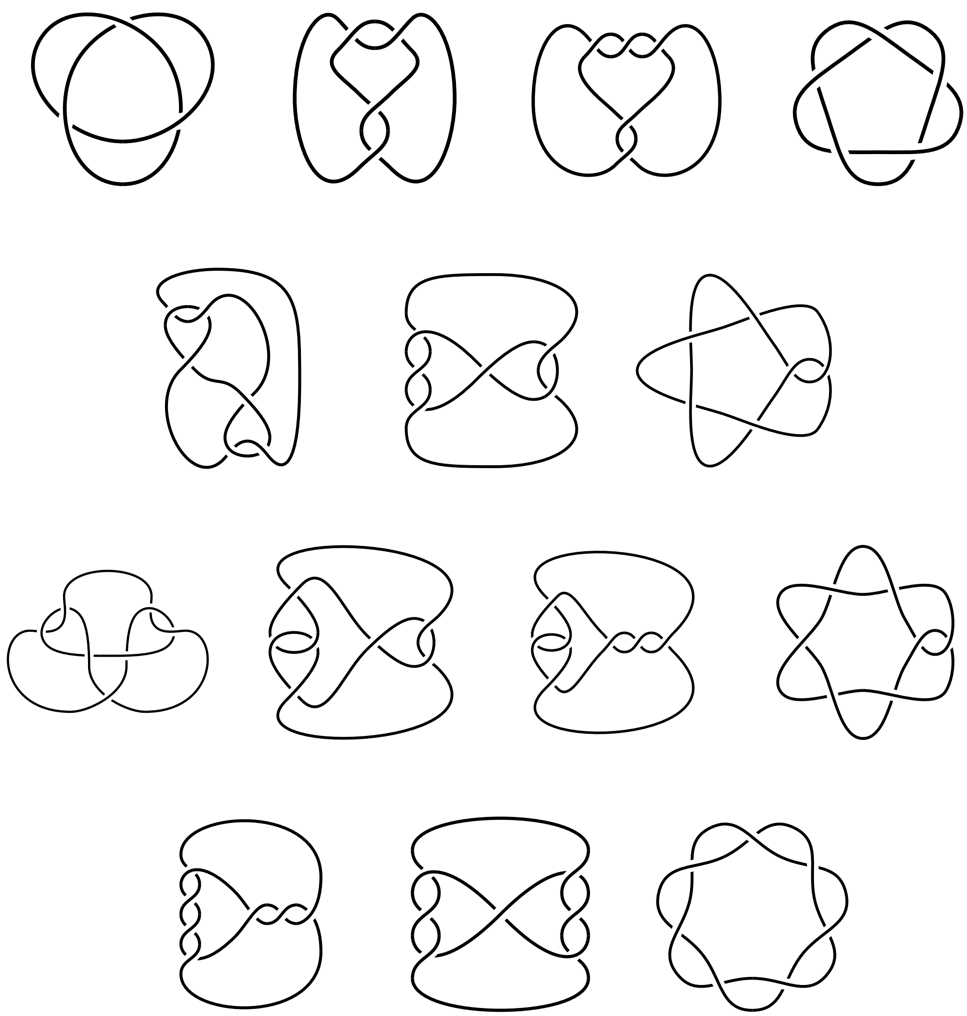
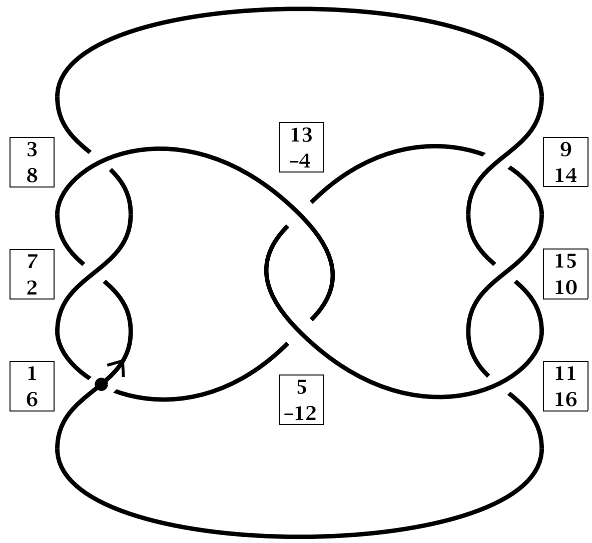


Table 1.

Exercise 1. Show that the fifth knot of Table 1 is amphicheiral.

Exercise 2. Invent an amphicheiral knot with 8 crossings.

It's often convenient to have a notation for knots. Here's a useful code for knot diagrams, based on an idea of **C.F. Gauss** and refined by **C.H. Dowker**:



1	3	5	7	9	11	13	15
6	8	−12	2	14	16	−4	10

Exercise 3. Find the minimal Dowker sequences for the first four knots of Table 1.

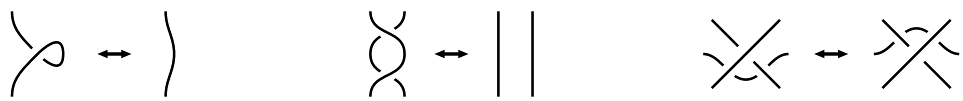
A basic problem of knot theory is the following: given knots K_1, K_2, how can we decide whether or not they are equivalent?

Typically, if we suspect that $K_1 \sim K_2$, we just have to find a way of deforming K_1 to K_2. However, if we suspect that $K_1 \nsim K_2$, we can't prove this by trying to deform K_1 to K_2 for a few hours and failing to do so. The notable 19th century Scottish physicist **P.G. Tait** was an avid tabulator of knots, but had the insight to recognize that he didn't have the means of distinguishing knots rigorously. He wrote: "... though I have grouped together many widely different but equivalent forms, I cannot be *absolutely* certain that all those groups are essentially different one from another."

A rigorous treatment of knots had to wait until **H. Poincaré** initiated the subject of topology in the early 20th century. The first proof that the right-hand and left-hand trefoils were inequivalent was published by **M. Dehn** in 1914.

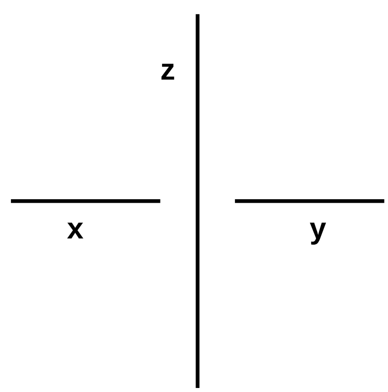
A standard method for dealing with difficult topological problems is to convert the problem to one of *algebra*. Hopefully the algebraic problem is easier, and solving it gives information on the original topological problem. In particular, there are various ways of assigning to each knot an algebraic object such as a *polynomial* or *group*. If, for example, we find that knots K_1, K_2 have distinct polynomials, then we may infer that K_1, K_2 are inequivalent. In this way, the polynomial behaves much like a reagent in chemistry. However, as with reagents, if the two polynomials turn out to be the same, the test has failed and we're none the wiser.

Knot colorings. In the 1930's, **K. Reidemeister** proved a very useful theorem. He proved that if two diagrams represent equivalent knots, then they're related via a sequence of finitely many *Reidemeister moves*:



Type I Type II Type III

The Reidemeister moves aren't useful for checking directly whether two knots are equivalent, but we can use them to construct algebraic invariants. Let n be a positive integer; in practice we need n to be an odd number ≥ 3. Note that the little gaps that indicate underpasses in a knot diagram separate the knot projection into arc segments. We say that a diagram *admits an n-coloring* if we can "label" each segment with an integer between 0 and $n-1$ inclusive, so that (i) at least two integers ("colors") are used overall, and (ii) at each crossing the following condition holds:



$$x + y - 2z \equiv 0 \pmod{n}$$

Exercise 4. Show that n-colorability of a diagram is invariant under the three Reidemester moves.

It follows that n-colorability is a knot invariant. If we have a diagram that can be n-colored for given n, and another that can't be n-colored, then the diagrams represent inequivalent knots.

This invariant is surprisingly effective: in particular all knots up to 7 crossings can be distinguished by this invariant, with the proviso that mirror images can't be distinguished. The effectiveness begins to wane thereafter, however.

Exercise 5. Find the smallest n for which the first 6-crossing knot illustrated on p.1 can be n-colored. *Hint:* use the diagram to set up a simple equation in n.

In the 1920's, **J.W. Alexander** discovered a *polynomial invariant* of knots. This polynomial is traditionally denoted $\Delta(t)$, and is the determinant of a matrix (whose entries are polynomials) obtainable from a knot diagram. It's related to n-colorability, in that $\Delta(-1) = n$ if and only if the knot is $|n|$-colorable. The Alexander polynomial can't distinguish between mirror images.

In 1984, **V.F.R. Jones** discovered a polynomial invariant $V(t)$ that opened an entirely new branch of knot theory. $V(t)$ is *very good* at distinguishing mirror images. For example,

$$V_{\text{RH trefoil}} = t + t^3 - t^4 \quad , \quad V_{\text{LH trefoil}} = t^{-1} + t^{-3} - t^{-4} \ .$$

There are some connections between $V(t)$ and $\Delta(t)$, in particular $V(-1) = \Delta(-1)$.

Following Jones's discovery, a plethora of new polynomial invariants was discovered, including the **HOMFLY** and **Kauffman 2-variable** polynomials. Although these new polynomials are quite powerful and can be used to prove general facts concerning knots, they're still not well understood from a geometric point of view. They appear to be more akin to *statistical mechanics* and *quantum field theory*.

Shortly we'll be able to put n-colorability into a deeper context, involving the *fundamental group of the knot complement*.