

Hyperbolic geometry 2

The prominent nineteenth century mathematician **Felix Klein** realized that geometries are *characterized by their isometries*. We need to familiarize ourselves with the isometries of \mathbb{H}^2 . Note that the set of all isometries of any metric space forms a *group* under the binary operation of composition.

In Euclidean geometry, isometries fall into two basic classes: (i) *direct* or *orientation-preserving isometries* (rotations, translations) and (ii) *indirect* or *orientation-reversing isometries* (reflections, glide reflections), depending whether the isometry preserves or reverses handedness.

◦	direct	indirect
direct	direct	indirect
indirect	indirect	direct

Table showing how the type of an isometry behaves under composition

It's convenient to regard the points of the Cartesian plane as *complex numbers*. Thus \mathbb{H}^2 consists of all complex numbers with positive imaginary part.

We've already seen one example of an isometry, namely inversion in a geodesic. If the geodesic is taken to be the y -axis, the effect of this isometry is simply to negate the x -coordinate of a point, *i.e.* its real part. We may write this isometry as

$$\rho_0 : x + iy \mapsto -x + iy = -(x - iy) = -(\overline{x + iy}) ,$$

or more simply as

$$\rho_0 : z \mapsto -\bar{z} .$$

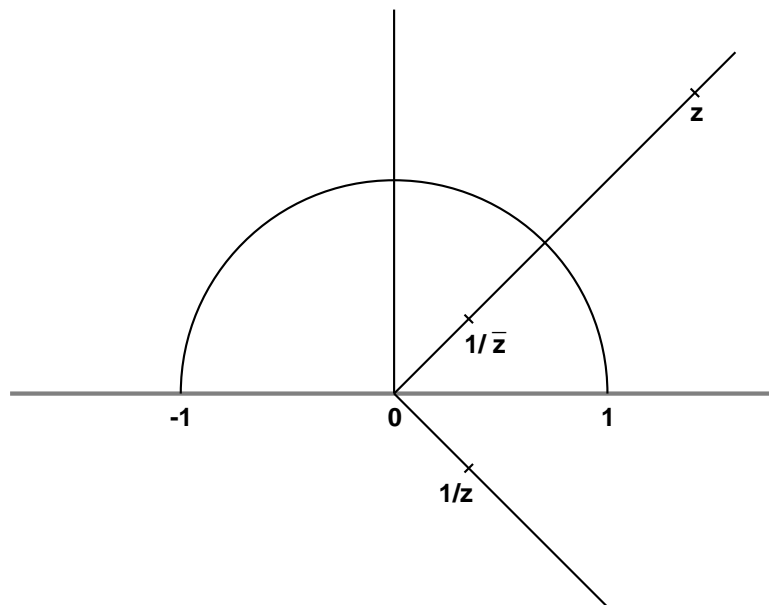
Slightly more generally, we may invert about the geodesic $x = a$, obtaining

$$\rho_a : z \mapsto -(\overline{z - a}) + a = -(\bar{z} - \bar{a}) + a = -\bar{z} + 2a .$$

Exercise 1. Let a, b be distinct real numbers, and let ρ_a, ρ_b be inversions in $x = a, x = b$ respectively, as above. Do ρ_a, ρ_b commute? Let $\tau = \rho_a \circ \rho_b$. Find an expression for $\tau(z)$. Which points on the boundary of \mathbb{H}^2 are fixed by τ ?

It turns out that all isometries of \mathbb{H}^2 can be obtained by composing inversions in geodesics. An isometry is *direct* if it is the composition of an even number of inversions, and *indirect* if it is the composition of an odd number of inversions. This is completely analogous to the situation in Euclidean plane geometry, where every isometry is obtainable by composing reflections.

So far we haven't investigated inversion in a semicircular geodesic. Let's take the geodesic with radius 1, centred at the origin; let σ be inversion in this geodesic.



It's easily verified (see figure) that σ acts on \mathbb{H}^2 as follows:

$$\sigma : z \mapsto \frac{1}{\bar{z}} .$$

Exercise 2. Find an expression for $(\rho_a \circ \sigma)(z)$. For which real numbers a do ρ_a, σ commute?

Exercise 3. Let a be a real number. Find all fixed points of the isometry $\rho_a \circ \sigma$, in \mathbb{H}^2 and on the boundary. (*Hint: experiment with some values of a ; it turns out that there are three cases to consider.*)

Exercise 4. Let σ be as above, and let ν be inversion in the semicircular geodesic of radius 2 and centre $(0, 0)$. Describe the effect of $\sigma \circ \nu$ geometrically.

By repeatedly composing isometries equal to σ or equal to ρ_a for some $a \in \mathbb{R}$, we can obtain any direct isometry of form

$$z \mapsto \frac{pz + q}{rz + s} \quad (ps - qr > 0) ,$$

and any indirect isometry of form

$$z \mapsto \frac{p\bar{z} + q}{r\bar{z} + s} \quad (ps - qr < 0) .$$

Direct isometries of form $z \mapsto pz$ ($p > 0$) are *dilations* centred at $(0, 0)$. Note that they fix two points on the boundary, namely the origin and ∞ , but don't fix any point in \mathbb{H}^2 . In general, direct isometries that fix two points on the boundary (and no points in \mathbb{H}^2) are called *hyperbolic isometries* or *hyperbolic translations*. They move points along the geodesic joining the two fixed points.

Direct isometries of form $z \mapsto z + q$ fix ∞ , but no other point on the boundary or in \mathbb{H}^2 . In general, direct isometries that fix one point on the boundary (and no points in \mathbb{H}^2) are called *parabolic isometries*.

Direct isometries that fix a single point in \mathbb{H}^2 are called *elliptic isometries*. They're analogous to rotations in Euclidean plane geometry. Elliptics don't look much like rotations in the upper half plane model, but they do look somewhat like rotations in the Poincaré disk model.

Let's take another look at

$$z \mapsto \frac{pz + q}{rz + s} \quad (p, q, r, s \in \mathbb{R}, ps - qr > 0) .$$

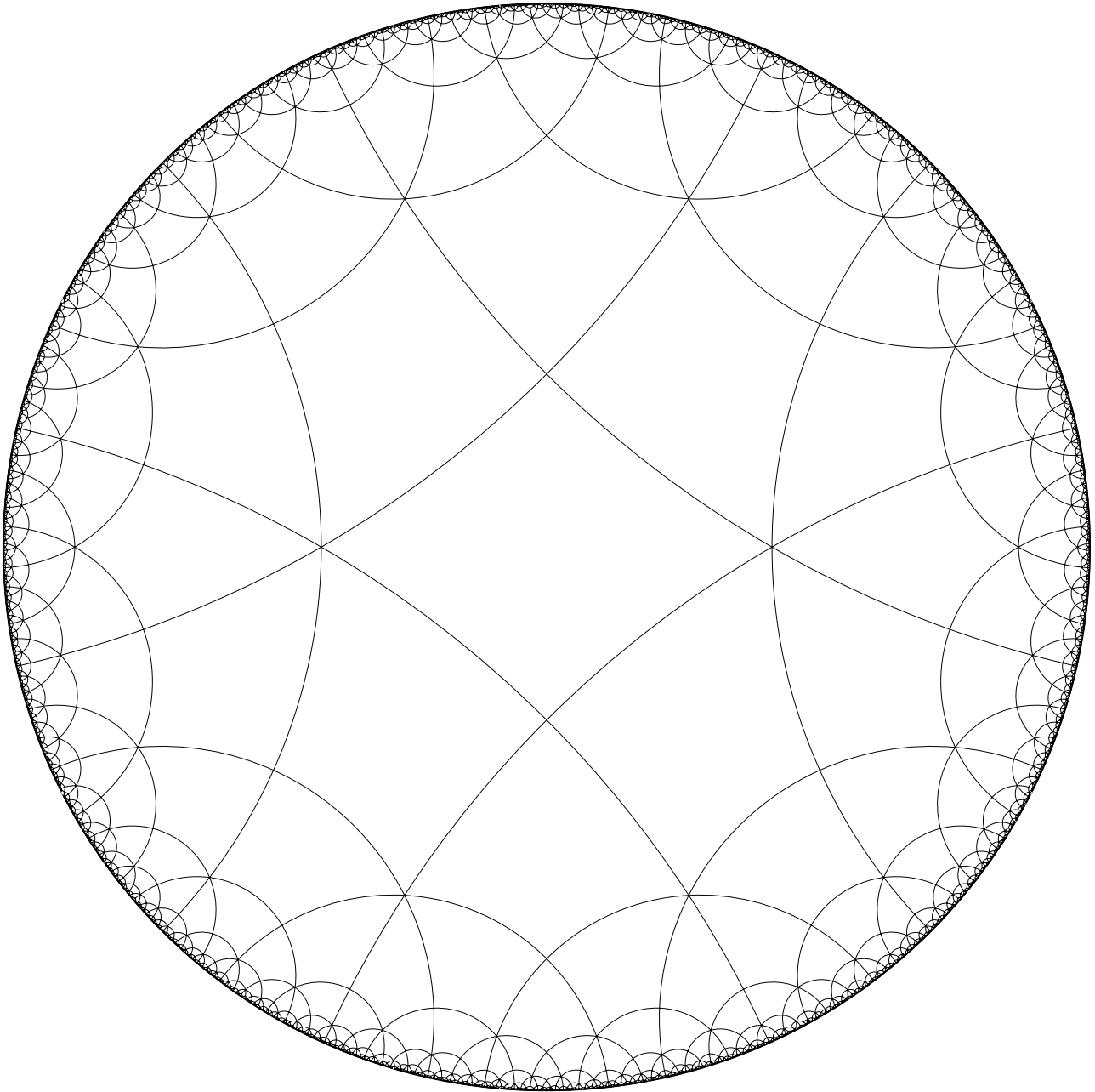
The real numbers p, q, r, s fit neatly into a 2×2 matrix; considering the complex number z as $z/1$, we can rewrite this as

$$\begin{bmatrix} z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} pz + q \\ rz + s \end{bmatrix} \quad \left(p, q, r, s \in \mathbb{R}, \det \begin{bmatrix} p & q \\ r & s \end{bmatrix} > 0 \right) .$$

Exercise 5. Let w_1, w_2 be complex numbers such that $z = w_1/w_2$. Show that it makes sense to find the image of z under the isometry of the previous paragraph by multiplying

the 2×2 matrix $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ by the 2×1 matrix $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

How can you use this matrix formulation to find the image of ∞ under the isometry?



Another tiling of the Poincaré disk. See if you can figure out its symmetries.